

UNIT-1

THE RANDOM VARIABLES

(10)

Review of probability theory:

Sample space:

The set of all possible outcomes in any experiment is called sample space.

Ex: Tossing a coin, throwing a dice, from a deck of pack of cards.

Event:

The set of possible outcomes in an experiment.

Eg: Getting odd number.

Independent events:

A set of events is said to be independent if occurs of any one of them doesn't depend on the occurs or not-occurs of the others.

→ doesn't depend on other events.

Mutually exclusive events:

If two sets have no common outcomes, they are called mutually exclusive events.

$$P(A \cup B) = P(A) + P(B)$$

If $P(A)$ and $P(B)$ are independent events, then

$$P(A \cup B) = P(A) \cdot P(B)$$

Definition of probability:

It is the ratio of total no. of favourable outcomes to total no. of possible outcomes.

$$P(A) = \frac{n(A)}{n(S)}$$

Properties (Axiomatic Conditions):

- 1) probability always lies between 0 and 1. $[0 \leq P(A) \leq 1]$
- 2) The sum of all probabilities $= 1$

$$\sum_{n=1}^N a_n = 1$$

- 3) The probability of not getting events is indicating $\bar{A}$ is

$$P(\bar{A}) = 1 - P(A)$$

Joint probability:

The probability of $A \cap B$ is called joint probability for two events A and B which is intersect in the sample space.

Algebra of set of properties:

- 1) $A \cup A = A$
 - 2) $A \cap A = A$
- } Independent law

Associative law:

$$* (A \cup B) \cup C = A \cup (B \cup C)$$

$$* (A \cap B) \cap C = A \cap (B \cap C)$$

commutative law:

$$* A \cup B = B \cup A$$

$$* A \cap B = B \cap A$$

Distributive law:

$$* A \cup \emptyset = A$$

$$* A \cap \emptyset = \emptyset$$

$$* A \cup \emptyset = A$$

$$* A \cap \emptyset = \emptyset$$

complement law:

$$* A \cup \bar{A} = S$$

$$* A \cap \bar{A} = \phi$$

Random Variable: * It is denoted by 'X'.

→ A Random variable is a real valued function defined on sample space associated with an experiment.

Conditions:

→ A function said to be random variable, acts if it satisfies the following conditions.

1) The function must be defined on sample space i.e., domain of the function must be ^{some} sample space.

2) It should be a real valued function i.e., codomain of function should be set of real numbers.

Types of Random variables:

There are ~~two~~ ^{three} types of Random variables. They are

(1) continuous

(2) Discrete

(3) Mixed

(1) continuous Random variable:

→ A Random variable which has an infinite no. of values is called continuous Random variable.

Ex: length, volume, time.

(2) Discrete Random variable:

→ A Random variable 'X' has finite range of values is called Discrete Random variable.

(3) Mixed

Ex: Tossing a coin.

(3) Mixed Random Variable

→ A Random variable which has combination of continuous and discrete Random variable.

probability distribution function (3) cumulative distribution function

* It describes the probab behaviour of probability.

$$F_X(x) = P(X \leq x), \text{ where } -\infty < x < \infty$$

Example: calculate cumulative distributive function

$$X = 0, 1, 2, 3, P(X) = 1$$

$$P(X) = \frac{1}{8} \quad \frac{3}{8} \quad \frac{3}{8} \quad \frac{1}{8}$$

Sol: $F_X(x) = P(X \leq x)$

$$P(X \leq x_1) = \frac{1}{8}$$

$$P(X \leq x_2) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$$

$$P(X \leq x_3) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8}$$

$$P(X \leq x_4) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$$

Types of probability distribution function:

(1) Discrete probability distribution function:

If 'X' is a Random variable then the probability of Random variable for 'i' events then the expression can be given as,

$$F_X(x_i) = \sum_{i=1}^N P(x_i) u(x - x_i)$$

↓
Unit step function.

Example:

$$X = 1 \quad 2 \quad 3 \quad 4$$

$$P(x_i) = \frac{1}{8} \quad \frac{3}{8} \quad \frac{3}{8} \quad \frac{1}{8}$$

$$F_X(x_i) = \sum_{i=1}^N P(x_i) \cdot u(x - x_i)$$

$$= \frac{1}{8} u(x-0) + \frac{3}{8} u(x-2) + \frac{3}{8} u(x-3) + \frac{1}{8} u(x-4)$$

(2) continuous probability distribution function:

$$F_X(x) = \int_{-\infty}^x f_X(x) dx$$

* In a given experiment if a random variable 'x' has elements 'x' which is equal to $x = \{1, 2, 3, 4, 5, 6\}$ with their probabilities $P(x) = \frac{2}{36}, \frac{8}{36}, \frac{12}{36}, \frac{7}{36}, \frac{5}{36}, \frac{2}{36}$. Find the cumulative distributive function (a) probability distributive function.

Sol: $x_i = \{1, 2, 3, 4, 5, 6\}$

$$P(x_i) = \frac{2}{36}, \frac{8}{36}, \frac{12}{36}, \frac{7}{36}, \frac{5}{36}, \frac{2}{36}$$

$$F_X(x) = P(X \leq x)$$

$$P(X \leq x_1) = \frac{2}{36}$$

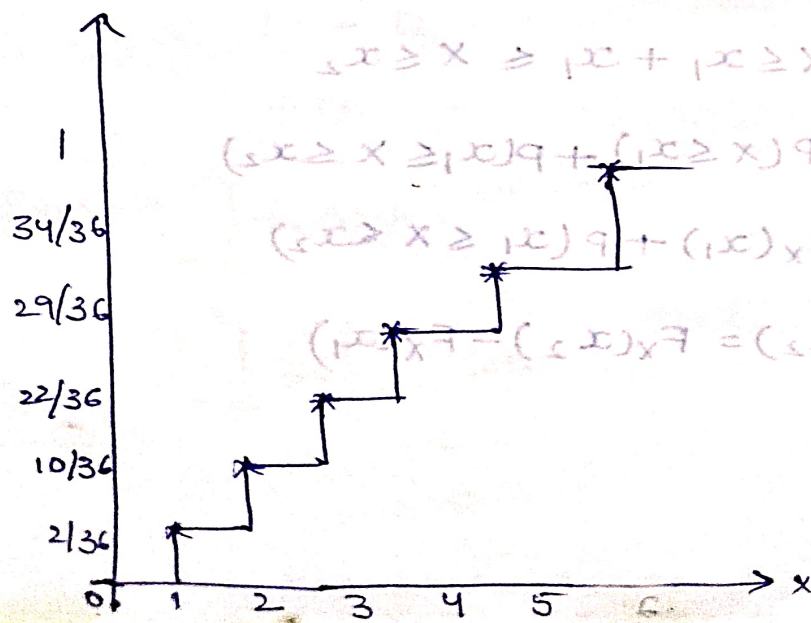
$$P(X \leq x_2) = \frac{2}{36} + \frac{8}{36} = \frac{10}{36}$$

$$P(X \leq x_3) = \frac{2}{36} + \frac{8}{36} + \frac{12}{36} = \frac{22}{36}$$

$$P(X \leq x_4) = \frac{2}{36} + \frac{8}{36} + \frac{12}{36} + \frac{7}{36} = \frac{29}{36}$$

$$P(X \leq x_5) = \frac{2}{36} + \frac{8}{36} + \frac{12}{36} + \frac{7}{36} + \frac{5}{36} = \frac{34}{36}$$

$$P(X \leq x_6) = \frac{2}{36} + \frac{8}{36} + \frac{12}{36} + \frac{7}{36} + \frac{5}{36} + \frac{2}{36} = 1$$



properties of probability distribution function:

$$1) F_X(-\infty) = 0$$

$$2) F_X(x) = P(X \leq x)$$

$$3) F_X(-\infty) = P(X \leq -\infty)$$

$$\boxed{F_X(-\infty) = 0}$$

$$2) F_X(\infty) = 1$$

$$F_X(x) = P(X \leq x)$$

$$= P(X \leq \infty) = P(S)$$

$$\boxed{F_X(\infty) = 1}$$

3) C.D.F bounded in between 0 & 1 $0 \leq F_X(x) \leq 1$

$$F_X(-\infty) \leq x \leq F_X(\infty)$$

$$0 \leq x \leq 1$$

4) $x_1 < x_2$ then $F_X(x_1) \leq F_X(x_2)$

$$x \leq x_1 \quad x \leq x_2 \quad \frac{1}{36} = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = (x \geq x_1) \cap (x \geq x_2)$$

$$P(X \leq x_1) \leq P(X \leq x_2) \quad \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = (x \geq x_1) \cap (x \geq x_2)$$

$$F_X(x_1) \leq F_X(x_2) \quad \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = (x \geq x_1) \cap (x \geq x_2)$$

5) probability $P(x_1 < X < x_2) = F_X(x_2) - F_X(x_1)$

$$P(x_1 < X < x_2) = F_X(x_2) - F_X(x_1)$$

$$x \leq x_2 = x \leq x_1 + x_1 \leq x \leq x_2$$

$$P(X \leq x_2) = P(X \leq x_1) + P(x_1 \leq X \leq x_2)$$

$$F_X(x_2) = F_X(x_1) + P(x_1 \leq X \leq x_2)$$

$$P(x_1 \leq X \leq x_2) = F_X(x_2) - F_X(x_1)$$

$$F_X(x) = P(X \leq x)$$

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$$P(x_1 < X < x_2) = P(x_1 \leq X \leq x_2)$$

$$P(x_1 < X < x_2) = P(X \leq x_2) - P(X \leq x_1) = F_X(x_2) - F_X(x_1)$$

$$P(x_1 < X < x_2) = P(X \leq x_2) - P(X \leq x_1)$$

$$\frac{1}{36} = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = (x \geq x_1) \cap (x \geq x_2)$$

$$\frac{1}{36} = (x \geq x_1) \cap (x \geq x_2)$$

$$\frac{1}{36} = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = (x \geq x_1) \cap (x \geq x_2)$$

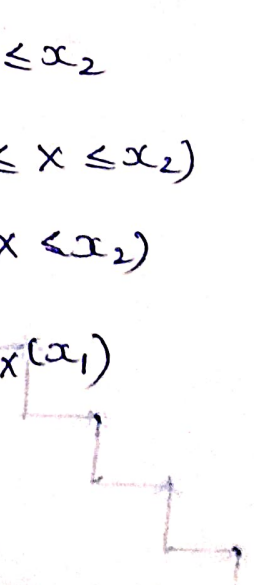
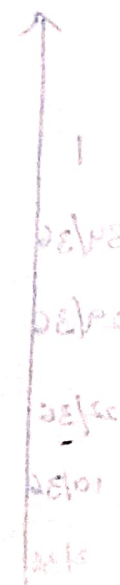
$$\frac{1}{36} = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = (x \geq x_1) \cap (x \geq x_2)$$

$$\frac{1}{36} + \frac{1}{36} + \frac{1}{36} = (x \geq x_1) \cap (x \geq x_2)$$

$$\frac{1}{36} + \frac{1}{36} + \frac{1}{36} = (x \geq x_1) \cap (x \geq x_2)$$

$$\frac{1}{36} + \frac{1}{36} + \frac{1}{36} = (x \geq x_1) \cap (x \geq x_2)$$

$$\frac{1}{36} + \frac{1}{36} + \frac{1}{36} = (x \geq x_1) \cap (x \geq x_2)$$



$$F(x) = 1 - F_X(x)$$

$$S = X > x + X \leq x$$

$$P(S) = P(X > x) + P(X \leq x)$$

$$1 = P(X > x) + F_X(x)$$

$$P(X > x) = 1 - F_X(x)$$

* Probability Density Function:

→ The PDF is always non-zero quantity. for all values of x .

→ We know that, probability distribution function always lies in between 0 and 1.

→ So, the derivative of distributive function always positive.

→ Area under PDF function is unity. i.e. $\int_{-\infty}^{\infty} f_X(x) dx = 1$

$$\int_{-\infty}^{\infty} f_X(x) dx = 1$$

$$\int_{-\infty}^{\infty} \frac{d}{dx} [F_X(x)] dx$$

$$[F_X(x)]_{-\infty}^{\infty}$$

$$= F_X(+\infty) - F_X(-\infty)$$

$$1 - 0$$

$$= 1$$

$$\rightarrow F_X(x) = \int_{-\infty}^x f_X(x) dx$$

$$= \int_{-\infty}^x \frac{d}{dx} [F_X(x)] dx$$

$$= [F_X(x)]_{-\infty}^x$$

$$= F_X(x) - F_X(-\infty)$$

→ probability of x lies in between x_1 and x_2 , $P(x_1 < x < x_2)$

$$\begin{aligned} P(x_1 < x < x_2) &= \int_{x_1}^{x_2} f_x(x) dx \\ &= \int_{x_1}^{x_2} \frac{d}{dx} [F_x(x)] dx \\ &= [F_x(x)]_{x_1}^{x_2} \\ &= F_x(x_2) - F_x(x_1) \end{aligned}$$

$$= P(x_1 < x < x_2)$$

For discrete Random Variables: derivative of step function is impulse function.

$$\frac{d}{dx} [F_x(x)] = \sum_{i=0}^N P(x_i) \frac{d}{dx} (u(x-x_i))$$

$$f_x(x_i) = \sum_{i=0}^N P(x_i) \delta(x-x_i)$$

* PDF is given by the expression $f_x(x) = ae^{-b|x|}$. Where x is a random variable and x varies from $-\infty$ to ∞ .

Determine the following.

(i) Relationship b/w a and b .

(ii) CDF

(iii) probability that outcomes lies in between 1 and 2.

Sol:

$$f_x(x) = \int_{-\infty}^{\infty} f_x(x) dx \quad \text{Given, } f_x(x) = ae^{-b|x|}$$

$$f_x(x) = \begin{cases} ae^{-bx} & x \geq 0 \\ ae^{bx} & x < 0 \end{cases}$$

Relation b/w a and b :

$$\text{WKT, } f_x(x) = \int_{-\infty}^{\infty} f_x(x) dx = 1$$

$$= \int_{-\infty}^0 ae^{bx} dx + \int_0^{\infty} ae^{-bx} dx = 1$$

$$= \int_{-\infty}^0 ae^{bx} dx + \int_0^{\infty} ae^{-bx} dx$$

$$= \left[\frac{ae^{bx}}{b} \right]_{-\infty}^0 + \left[\frac{ae^{-bx}}{-b} \right]_0^{\infty} = 1$$

$$= \cancel{\frac{ae^{b(-\infty)}}{b}} - \cancel{\frac{ae^{b(\infty)}}{b}}$$

$$= \frac{a}{b}[1-0] + \frac{a}{b}[0-1] = 1$$

$$= \frac{a}{b} + \frac{a}{b} = 1$$

$$= \frac{2a}{b} = 1$$

$$2a = b$$

$$\frac{a}{b} = \frac{1}{2}$$

* Find the CDF

$$F_X(x) = \int_{-\infty}^x f_X(x) dx$$

$$F_X(x) = \int_{-\infty}^0 ae^{bx} dx + \int_0^x ae^{-bx} dx$$

$$= \left[\frac{ae^{bx}}{b} \right]_{-\infty}^0 + \left[\frac{ae^{-bx}}{-b} \right]_0^x$$

$$= \frac{a}{b}[1-0] - \frac{a}{b}[e^{-bx} - 1]$$

$$= \frac{a}{b}[1 - e^{-bx} + 1]$$

$$= \frac{a}{b}[2 - e^{-bx}]$$

$$F_X(x) = \frac{a}{b}[2 - e^{-bx}]$$

$$(iii) P(x_1 < X < x_2) = \int_{x_1}^{x_2} f_X(x) dx$$

$$= \int_{x_1}^{x_2} ae^{-bx} dx = \left[\frac{ae^{-bx}}{-b} \right]_{x_1}^{x_2} = -\frac{a}{b}[e^{-bx_2} - e^{-bx_1}]$$

$$= \frac{1}{2} [e^{-b} - e^{-2b}]$$

$$P(1 < x < 2) = \frac{1}{2} [e^{-b} - e^{-2b}] //$$

$$* X = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$$

$$P(X) = K \quad 3K \quad 5K \quad 7K \quad 9K \quad 11K \quad 13K$$

(i) find the 'k' value

(ii) find the ^{probability} distribution and ^{probability} density function

Sol: (i)

sum of all probabilities = 1

$$\therefore K + 3K + 5K + 7K + 9K + 11K + 13K = 1$$

$$49K = 1$$

$$K = \frac{1}{49}$$

$$\text{Now, } X = \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}$$

$$P(X) = \frac{1}{49} \quad \frac{3}{49} \quad \frac{5}{49} \quad \frac{7}{49} \quad \frac{9}{49} \quad \frac{11}{49} \quad \frac{13}{49}$$

(ii) probability distribution function

$$F_X(x) = P(X \leq x_0)$$

$$P(X \leq x_1) = \frac{1}{49}$$

$$P(X \leq x_2) = \frac{1}{49} + \frac{3}{49} = \frac{4}{49}$$

$$P(X \leq x_3) = \frac{1}{49} + \frac{3}{49} + \frac{5}{49} = \frac{9}{49}$$

$$P(X \leq x_4) = \frac{1}{49} + \frac{3}{49} + \frac{5}{49} + \frac{7}{49} = \frac{16}{49}$$

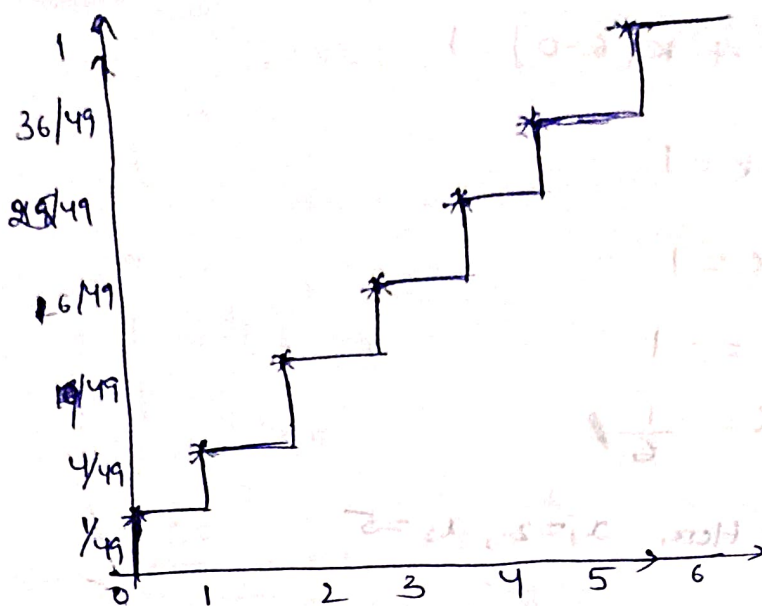
$$P(X \leq x_5) = \frac{1}{49} + \frac{3}{49} + \frac{5}{49} + \frac{7}{49} + \frac{9}{49} = \frac{25}{49}$$

$$P(X \leq x_6) = \frac{1}{49} + \frac{3}{49} + \frac{5}{49} + \frac{7}{49} + \frac{9}{49} + \frac{11}{49} = \frac{36}{49}$$

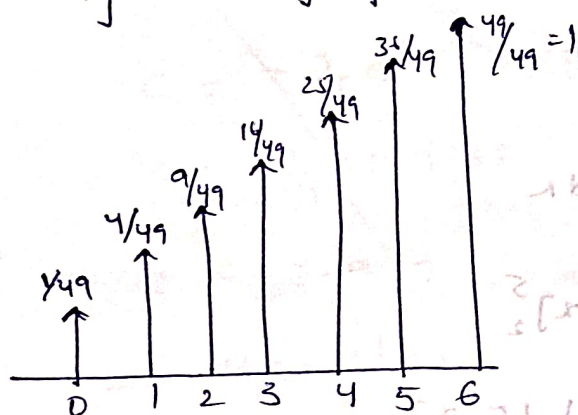
$$P(X \leq x_7) = \frac{1}{49} + \frac{3}{49} + \frac{5}{49} + \frac{7}{49} + \frac{9}{49} + \frac{11}{49} + \frac{13}{49} = \frac{49}{49}$$

$$= \frac{1}{49} u(x-0) + \frac{4}{49} u(x-1) + \frac{9}{49} u(x-2) + \frac{16}{49} u(x-3) + \frac{25}{49} u(x-4) + \frac{36}{49} u(x-5) +$$

$$u(x-6)$$



(iii) probability density function,



$$= \frac{1}{49} \delta(x-0) + \frac{4}{49} \delta(x-1) + \frac{9}{49} \delta(x-2) + \frac{16}{49} \delta(x-3) + \frac{25}{49} \delta(x-4) + \frac{36}{49} \delta(x-5) + 1 \delta(x-6).$$

* Let x be a constant random variable with density function $f_x(x) = \left(\frac{x}{9} + k\right)$, $(0 \leq x \leq 6)$ - (i) find the

value of k .

(ii) $p(2 \leq X \leq 5)$

Sol: WKT, $f_x(x) = \int_{-\infty}^{\infty} f_x(x) dx = 1$

$$\int_0^6 \left(\frac{x}{9} + k\right) dx = 1$$

$$\int_0^6 \frac{x}{9} dx + \int_0^6 k dx = 1$$

$$\frac{1}{9} \left[\frac{x^2}{2}\right]_0^6 + k[x]_0^6 = 1$$

$$= \frac{1}{18} [36-0] + K [6-0] = 1$$

$$\frac{36}{18} + 6K = 1$$

$$2 + 6K = 1$$

$$6K = -1$$

$$K = -\frac{1}{6}$$

(ii) $P(2 \leq X \leq 5) = ?$ Here, $x_1 = 2, x_2 = 5$

$$\int_{x_1}^{x_2} f_X(x) dx$$

$$= \int_2^5 \left[\frac{x}{9} + \left(-\frac{1}{6}\right) \right] dx$$

$$= \int_2^5 \frac{x}{9} dx - \frac{1}{6} \int_2^5 dx$$

$$= \frac{1}{9} \left[\frac{x^2}{2} \right]_2^5 - \frac{1}{6} [x]_2^5$$

$$= \frac{1}{9} \left[\frac{25}{2} - 4 \right] - \frac{1}{6} [5-2]$$

$$= \frac{21}{18} - \frac{1}{2} = \frac{7}{6} - \frac{3}{6} = \frac{4}{6} = \frac{2}{3}$$

$$= \frac{7-3}{6} = \frac{4}{6} = \frac{2}{3}$$

$$= \frac{2}{3}$$

$$= \frac{4}{6}$$

$$P(2 \leq X \leq 5) = \frac{2}{3}$$

* If the probability density function of random variable is given by $f_X(x) = c e^{\left(-\frac{x}{4}\right)}$ ($0 < x < 1$). Find the value of c . and $F_X(0.5)$.

$$1 = \int_0^1 c e^{-\frac{x}{4}} dx$$

$$1 = c \left[-4 e^{-\frac{x}{4}} \right]_0^1$$

Sol:
(b)

WKT,

$$f_x(x) = \int_{-\infty}^{\infty} f_x(x) dx = 1$$

$$= \int_0^1 c e^{-\frac{x}{4}} dx = 1$$

$$= \left[\frac{c e^{-\frac{x}{4}}}{-\frac{1}{4}} \right]_0^1 = 1$$

$$= \frac{c e^{-\frac{1}{4}}}{-\frac{1}{4}} = 1, \quad \frac{c e^{-\frac{1}{4}}}{-\frac{1}{4}} - c e^0 = 1$$

$$c e^{-\frac{1}{4}} = -\frac{1}{4}$$

$$c = -\frac{1}{4} e^{-\frac{1}{4}} \quad [e^{-1/4} = 0.7788]$$

$$c = -\frac{1}{4} \times 0.7788$$

$$c = -0.1947$$

$$= -4 c e^{-\frac{1}{4}} = 1$$

$$c e^{-\frac{1}{4}} = -\frac{1}{4}$$

$$= -4 c [e^{-\frac{1}{4}} - e^0] = 1$$

$$-4 c [0.7788 - 1] = 1$$

$$-4 c = \frac{1}{-0.2212}$$

$$c = \frac{1}{0.8848}$$

$$c = 1.1301$$

ii). $F_x(0.5)$

WKT, $F_x(x) = \int_{-\infty}^x f_x(x) dx$

Here, $x = 0.5$

$$F_x(0.5) = \int_{-\infty}^{0.5} c e^{-\frac{x}{4}} dx$$

$$\begin{aligned}
 &= \int_{-\infty}^{\infty} c e^{(-x/4)} dx \\
 &= \int_{-\infty}^{0.5} c e^{(x/4)} dx \\
 &= \int_{-\infty}^{0.5} (1.13) e^{-x/4} dx \\
 &= 1.13 \left[\frac{e^{-x/4}}{-1/4} \right]_{-\infty}^{0.5} \\
 &= -4.52 [e^{-0.5/4} - e^{\infty/4}] \\
 &= -4.52 [0.8824] \\
 &= -3.988
 \end{aligned}$$

* The probability density function of Random Variable 'x' is given by $f_x(x) = k$, ($a < x < b$), find the k value.

sa:

WKT, $\int_{-\infty}^{\infty} f_x(x) dx = 1$

$$\int_a^b k dx = 1$$

$$k[x]_a^b = 1 = [1 - 0.8824] \times 1.13$$

$$k[b-a] = 1 \Rightarrow \frac{1}{1.1176} = 0.8946$$

$$k = \frac{1}{b-a}$$

$$k = \frac{1}{b-a}$$

$$k = \frac{1}{b-a}$$

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

Ex. 2. If the probability density given by $f_x(x) = k(1-x^2)$. Find the k value, and CDFs ($0 \leq x < 1$)

Sol:

$$\int_{-\infty}^{\infty} f_x(x) = 1$$

$$\int_0^1 k(1-x^2) = 1$$

$$k \left[x - \frac{x^3}{3} \right]_0^1 = 1$$

$$k \left[1 - \frac{1}{3} \right] = 1$$

$$k \left[\frac{2}{3} \right] = 1$$

$$k = \frac{3}{2}$$

$$k = 1.5$$

(ii) CDF:

$$F_x(x) = \int_{-\infty}^x f_x(x) dx$$

$$= \int_0^x k(1-x^2) dx$$

$$= k \left[x - \frac{x^3}{3} \right]_0^x$$

$$= k \left[x - \frac{x^3}{3} \right]$$

$$= k \left[x - \frac{x^3}{3} \right]_0^x$$

$$= 1.5 \left[x - \frac{x^3}{3} \right]$$

$$= \frac{3}{2} \left[\frac{3x - x^3}{3} \right]$$

$$= \frac{3x - x^3}{2}$$

* The PDF of Random Variable $f_x(x) = k(e^{-ax})^a$ where 'a' is a positive constant. Determine the k value.

Sol:

$$\int_{-\infty}^{\infty} f_x(x) dx = 1$$

$$\int_0^{\infty} k e^{-ax} dx = 1$$

$$k \left[\frac{e^{-ax}}{-a} \right]_0^{\infty} = 1$$

$$\frac{k}{-a} (e^{-\infty} - e^0) = 1$$

$$\frac{k}{-a} (-1) = 1$$

$$k = +a$$

* If the probability density function is $f_x(x) = x$ for $0 < x < 1$, $(2-x)$ for $1 < x < 2$.

→ Find the probability of Random variables lies between

0.2 to 0.8.

→ 0.6 to 1.2

Sol:

WKT, $\int_{-\infty}^{\infty} f_x(x) dx = 1$

$$\int_0^1 x dx + \int_1^2 (2-x) dx = 1$$

$$\left[\frac{x^2}{2} \right]_0^1 + \left[2x - \frac{x^2}{2} \right]_1^2 = 1$$

$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} f_x(x) dx$$

$$F_x(x) = \int_{0.2}^{0.8} x dx$$

$$= \left[\frac{x^2}{2} \right]_{0.2}^{0.8}$$

$$= \frac{1}{2} [0.8^2 - 0.2^2]$$

$$= \frac{1}{2} [0.64 - 0.04]$$

$$F_x(x) = 0.3$$

$$f_x(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 < x < 2 \end{cases}$$

$$F_X(x) = \int_{0.6}^1 F_X(x) dx + \int_1^{1.2} F_X(x) dx$$

$$= \int_{0.6}^1 x dx + \int_1^{1.2} (2-x) dx$$

$$= \left[\frac{x^2}{2} \right]_{0.6}^1 + \left[2x - \frac{x^2}{2} \right]_1^{1.2}$$

$$= \left[\frac{1}{2} - \frac{0.6^2}{2} \right] + \left[2(1.2) - \frac{(1.2)^2}{2} - 2(1) + \frac{1}{2} \right]$$

$$= \left[\frac{1}{2} - \frac{0.36}{2} \right] + \left[2.4 - 0.72 - 2 + 0.5 \right]$$

$$= 0.5 - 0.18 + 2.4 - 0.72 - 2 + 0.5$$

$$= 0.5 //$$

Gaussian distribution and density functions:

Let 'x' be a random variable then it's Gaussian density function $f_X(x)$ can be defined as

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\mu)^2}{2\sigma_x^2}}$$

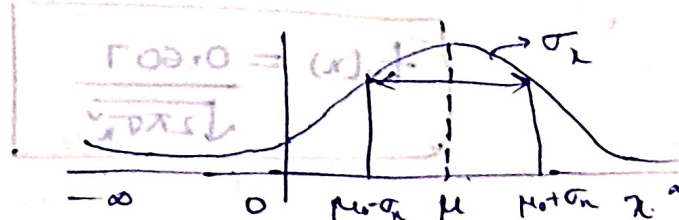
where, $\mu \Rightarrow$ Mean

$\sigma_x =$ standard deviation

$\sigma_x^2 =$ variance

The probability distribution function for the Gaussian Random variable is given as,

$$F_X(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\mu)^2}{2\sigma_x^2}}$$



where, $\sigma_x > 0$, $-\infty < \mu_x < \infty$

$-\infty < \mu_x < \infty$

Step 1: At $x = \mu$
Case-1:

$$f_x(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(\mu-\mu)^2}{2\sigma_x^2}}$$

$$f_x(x) = \frac{1}{\sqrt{2\pi}\sigma_x}$$

Case-2: At $x = \mu - \sigma_x$

$$f_x(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(\mu-\sigma_x-\mu)^2}{2\sigma_x^2}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{\sigma_x^2}{2\sigma_x^2}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_x} e^{-1/2}$$

$$f_x(x) = \frac{0.607}{\sqrt{2\pi}\sigma_x}$$

Case-3: At $x = \mu + \sigma_x$

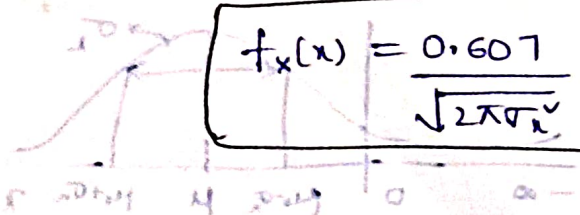
$$f_x(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(\mu+\sigma_x-\mu)^2}{2\sigma_x^2}}$$

$$f_x(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{\sigma_x^2}{2\sigma_x^2}}$$

$$f_x(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-1/2}$$

$$f_x(x) = \frac{0.607}{\sqrt{2\pi}\sigma_x}$$

$$f_x(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-\mu)^2}{2\sigma_x^2}}$$



If mean $(\mu_2 \text{ or } \mu) = 0$ and standard deviation $= 1$

$$F_X(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx$$

$$= \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{x^2}{2}} dx$$

Uniform density function:

It is defined as

$$f_X(x) = \frac{1}{b-a}, \quad a < x < b$$

Uniform distribution function:

$$\text{WKT, } F_X(x) = \int_{-\infty}^x f_X(x) dx$$

$$= \int_a^x \frac{1}{b-a} dx$$

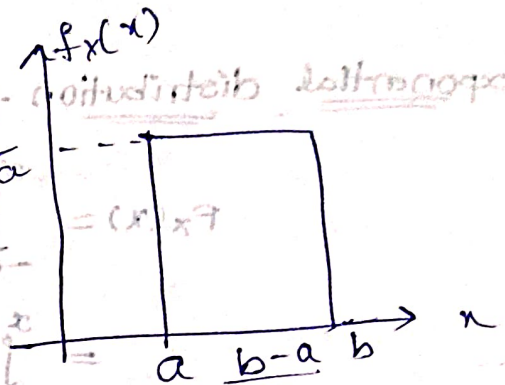
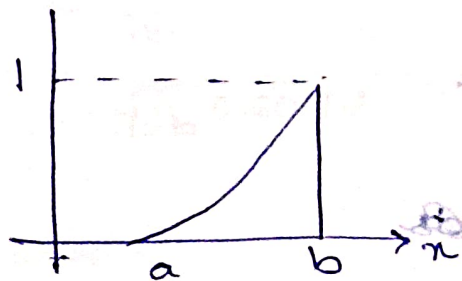
$$= \frac{1}{b-a} \int_a^x dx$$

$$= \frac{1}{b-a} [x]_a^x = \frac{1}{b-a} (x - a)$$

$$= \frac{x-a}{b-a}$$

$$F_X(x) = 1 \quad x > b$$

$$F_X(x) = \frac{x-a}{b-a}$$

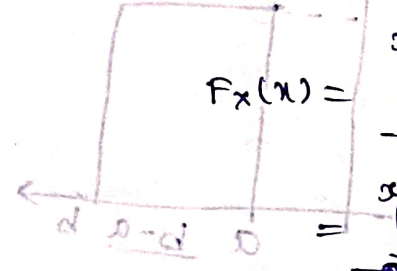


Exponential density function:

A continuous Random variable 'x' is said to be have exponential distribution with parameter ' λ ' (known as rate parameter) if 'x' has density function,

for 1 variable,
$$f_x(x) = \lambda e^{-\lambda x}, x > 0.$$
$$= 0, \text{ Otherwise.}$$

Exponential distribution function:


$$F_X(x) = \int_{-\infty}^x f_X(x) dx$$

$$= \int_0^x \lambda e^{-\lambda x} dx$$

$$= \int_0^x \lambda e^{-\lambda x} dx$$

$$= \left[\frac{\lambda e^{-\lambda x}}{-\lambda} \right]_0^x$$

$$= -\left(e^{-\lambda x} \right) + e^{-0}$$

$$F_X(x) = 1 - e^{-\lambda x}$$

For 2 variables,

$$f_x(x) = \frac{1}{b} e^{-\left(\frac{x-a}{b}\right)}, x > a$$

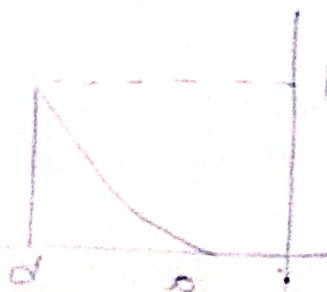
0, Otherwise

$$F_X(x) = \int_{-\infty}^x f_X(x) dx$$

$$= \int_a^x \frac{1}{b} e^{-\left(\frac{x-a}{b}\right)} dx$$

$$= \frac{1}{b} \left[\frac{e^{-\left(\frac{x-a}{b}\right)}}{-\frac{1}{b}} \right]_a^x$$

$$\frac{a-x}{a-d} = (x)x^7$$

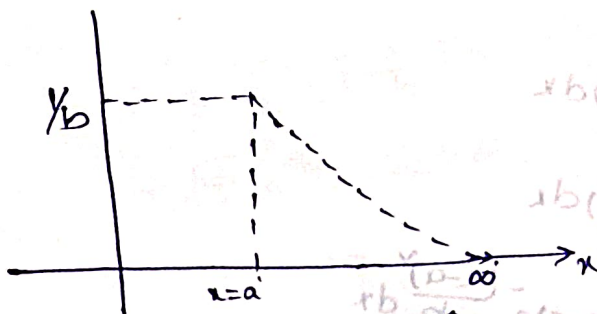


$$-\left[e^{-\left[\frac{x-a}{b}\right]}\right]_a^x$$

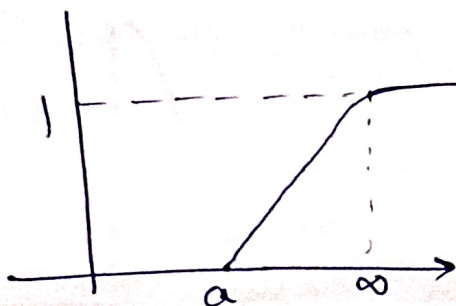
$$= -\left[e^{-\left[\frac{x-a}{b}\right]} - e^0\right]$$

$$F_X(x) = 1 - e^{-\left[\frac{x-a}{b}\right]}$$

Density



Distribution:



Maximum value for probability density function:

$$\frac{d}{dx} f_X(x) = 0$$

$$x = a + \sqrt{\frac{b}{2}}$$

$$f_X(x) = \frac{2}{b}(x-a)e^{-\frac{(x-a)}{b}}$$

$$= \frac{2}{b}\left(a + \sqrt{\frac{b}{2}} - a\right)e^{-\left[\frac{a + \sqrt{b/2} - a}{b}\right]}$$

$$= \frac{2}{b}\sqrt{\frac{b}{2}}e^{-\left(\frac{\sqrt{b/2}}{b}\right)}$$

$$= \frac{\sqrt{2}}{\sqrt{b}}e^{-1/2}$$

$$= \sqrt{\frac{2}{b}} \cdot 0.607 //$$

$$F_X(x) = 1 - e^{-\left[\frac{x-a}{b}\right]}$$

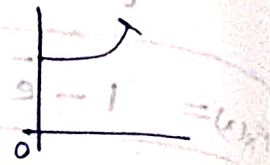
Rayley : Rayleigh :

Random variable x is defined as

$$f_x(x) = \frac{2}{b}(x-a)e^{-\frac{(x-a)^2}{b}}$$

$$x \geq a$$

$$x < a$$



Distribution function:

$$F_x(x) = \int_{-\infty}^x f_x(x) dx$$

$$F_x(x) = \int_a^x f_x(x) dx$$

$$F_x(x) = \int_a^x \frac{2}{b}(x-a)e^{-\frac{(x-a)^2}{b}} dx$$

let, $\frac{(x-a)^2}{b} = y$

$$\frac{d}{dx} \left[\frac{(x-a)^2}{b} \right] = \frac{dy}{dx}$$

$$\frac{2(x-a)}{b} = \frac{dy}{dx}$$

$$\frac{2(x-a)}{b} dx = dy$$

$$F_x(x) = \int_a^x \frac{2}{b}(x-a)e^{-\frac{(x-a)^2}{b}} dx$$

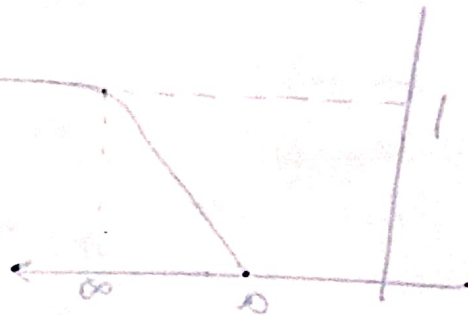
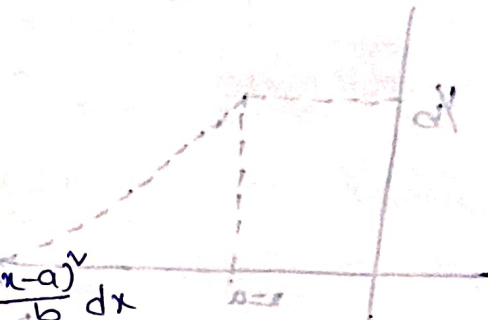
$$F_x(x) = - \left[e^{-\frac{(x-a)^2}{b}} \right]_a^x$$

$$= - \left[e^{-\frac{(x-a)^2}{b}} - e^{-\frac{(a-a)^2}{b}} \right]$$

$$= - \left[e^{-\frac{(x-a)^2}{b}} - 1 \right]$$

$$= 1 - e^{-\frac{(x-a)^2}{b}}$$

$$F_x(x) = 1 - e^{-\frac{(x-a)^2}{b}}$$



$$0 = (x-a) + \frac{b}{x-a}$$

$$\frac{d}{dx} \left[(x-a) + \frac{b}{x-a} \right] = 0$$

$$1 - \frac{b}{(x-a)^2} = 0$$

$$1 = \frac{b}{(x-a)^2}$$

Maximum value of probability density function:

$$\frac{d}{dx} f_x(x) = 0$$

$$x = a + \sqrt{\frac{b}{2}}$$

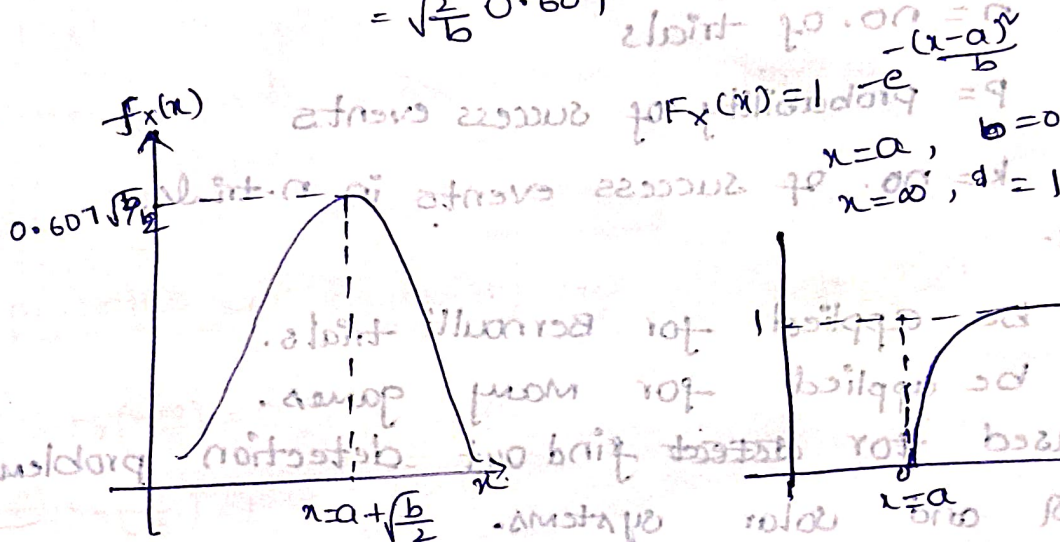
$$f_x(x) = \frac{2}{b} (x-a) e^{-\frac{(x-a)^2}{b}}$$

$$= \frac{2}{b} \left[a + \sqrt{\frac{b}{2}} - a \right] e^{-\frac{\left[a + \sqrt{\frac{b}{2}} - a \right]^2}{b}}$$

$$= \frac{2}{b} \sqrt{\frac{b}{2}} e^{-\left(\frac{\sqrt{b/2}}{b}\right)^2}$$

$$= \sqrt{\frac{2}{b}} e^{-1/2}$$

$$= \sqrt{\frac{2}{b}} 0.607$$



Binomial distribution and density function:

Let us consider an experiment which has only two outcomes where one outcome is called success and another outcome is failure.

→ Let this experiment is represented by no. of trials. These are nothing but "Bernoulli trials".

→ And let us consider probability of success is 'P' and failure is 'Q' each trial is same.

→ The trials are independent then probability of failure in each trial is 'q' where, $q = 1 - p$ $q = 1 - p$

→ The probability of ~~(k=0)~~... 'k' successes in n-trials given by probability function known as binomial probability density function.

$$f_x(x) = \sum_{k=0}^n n C_k P^k q^{n-k} \delta(n-k) \quad \text{where, } n C_k = \frac{n!}{k!(n-k)!}$$

$$F_x(x) = \sum_{k=0}^n n C_k P^k q^{n-k} u(n-k)$$

where, n = no. of trials

P = probability of success events

k = no. of success events in n-trials.

Applications:-

- It can be applied for Bernoulli trials.
- It can be applied for many games.
- It is used for ~~detect~~ find out detection problems in radar and solar systems.

Poisson distribution function; and density function:

$$f_x(x) = \sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} \delta(n-k)$$

where, $\lambda = np = \text{constant}$

n = no. of trials

P = probability of success events

k = no. of success events.

$$F_x(x) = \sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} u(n-k)$$

403, 425, 478, 4A0, 4F2, 4F3,

4I4, 4J7, 401, 4P3, 4Q2

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conditional probability:

consider, A and B are two events, the probability of A/B can find out, when event B is known,

[3]

If two events A and B , ~~are~~ the event B is known then the probability of A/B is $P(A/B) = \frac{P(A \cap B)}{P(B)}$

conditional probability distribution and density function:

consider A and B are two events, the probability of $P(A/B)$ can find out, when event B is known.

→ The conditional distribution function is

$$F_X(A/B) = P(X \leq A/B)$$

properties of ^{conditional} probability distribution function:

- 1) $F_X(-\frac{\infty}{B}) = 0$
- 2) $F_X(\frac{\infty}{B}) = 1$
- 3) $0 < F_X(\frac{x}{B}) \leq 1$
- 4) If $x_1 < x_2$, then $F_X(\frac{x_1}{B}) \leq F_X(\frac{x_2}{B})$
- 5) If $P(x_1 < X < x_2)$ ^{then} $F_X(\frac{x_2}{B}) - F_X(\frac{x_1}{B})$
- 6) If $P(X > x)$ then $= 1 - F_X(\frac{x}{B})$

properties of conditional probability density function:

→ The derivative of conditional distribution function is known as conditional probability density function.

Properties:

1) $f_X(x/B) \geq 0$

2) $\int_{-\infty}^{\infty} f_X(x/B) dx = 1$

3) $F_X(x/B) = \int_{-\infty}^x f_X(x/B) dx$

4) $P(x_1 < X < x_2) = \int_{x_1}^{x_2} f_X(x/B) dx$

→ total probability theorem

→ Baye's theorem

→ probability mass functions:

1) $F_X(-\infty/B) = 0$

2) $F_X(\infty/B) = 1$

3) $0 < F_X(x/B) \leq 1$

4) If $x_1 < x_2$, then $F_X(x_1/B) \leq F_X(x_2/B)$

5) If $P(x_1 < X < x_2) = F_X(x_2/B) - F_X(x_1/B)$

6) If $P(X > x) = 1 - F_X(x/B)$

Properties of conditional probability density function

The derivative of conditional probability density function

is known as conditional probability density function

UNIT-2 OPERATION ON SINGLE RANDOM VARIABLE

(MATHEMATICAL EXPECTATIONS)

→ Mathematical Expectations or Mean or Average or Statistical Average value.

→ consider the mean value or average value of random variable 'x' is called Mathematical Expectation of 'x'.

→ It is denoted with $\bar{x}$ or $E[x]$.

$$E[x] = \bar{x} = \int_{-\infty}^{\infty} x f_x(x) dx$$

* Find the mean value for uniform density function.

Sol:

$$\int_{-\infty}^{\infty} x \left[\frac{1}{b-a} \right] dx$$

$$\frac{1}{b-a} \int_a^b x dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b$$

$$= \frac{1}{2(b-a)} [x^2]_a^b$$

$$= \frac{b^2 - a^2}{2(b-a)}$$

$$= \frac{(b+a)(b-a)}{2(b-a)}$$

$$= \frac{b+a}{2}$$

$$E(x) = \frac{a+b}{2} = \bar{x}$$

Expected value of Random Variable for discrete function:

If 'x' is a discrete Random variable, set of elements like with $\{x_1, x_2, x_3, \dots, x_n\}$ with probabilities is $\{P(x_1), P(x_2), P(x_3), \dots, P(x_n)\}$ then the expected value for discrete function of discrete random variables is given as,

$$E(x) = \bar{x} = \sum_{i=1}^n x_i P(x_i)$$

Expected value for the function:

consider a Random variable 'x' with its probability density function $f_x(x)$.

→ If $g(x)$ be a function of random variable then the mean value can be defined as,

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f_x(x) dx$$

→ If 'x' is a discrete random variable then 'x' is defined as,

$$E[g(x)] = \sum_{i=1}^n g(x_i) P(x_i)$$

conditional expectation of random variables:

$$E\left[\frac{x}{B}\right] = \int_{-\infty}^{\infty} x \cdot f_x\left(\frac{x}{B}\right) dx$$

Mean Square Error value:

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f_x(x) dx$$

Properties of expected value:

- 1) If a Random variable is constant, then $E(x)$ always constant.

$$E(a) = \int_{-\infty}^{\infty} a \cdot f_x(x) dx = a \int_{-\infty}^{\infty} f_x(x) dx \quad E(a) = a$$
$$\boxed{E(ax) = a}$$

- 2) If a is any constant, then $\boxed{E(ax) = a \cdot E(x)}$

$$E(ax) = \int_{-\infty}^{\infty} ax \cdot f_x(x) dx$$
$$= a \int_{-\infty}^{\infty} x \cdot f_x(x) dx$$
$$= a E(x)$$

- 3) If a and b are any two constants, then,

$$\boxed{E(ax+b) = a[E(x)] + b}$$

$$E(ax+b) = \int_{-\infty}^{\infty} (ax+b) \cdot f_x(x) dx$$
$$= \int_{-\infty}^{\infty} ax \cdot f_x(x) dx + b \int_{-\infty}^{\infty} f_x(x) dx$$
$$= a \int_{-\infty}^{\infty} x \cdot f_x(x) dx + b[1]$$
$$= a E(x) + b$$

→ If x is a discrete random variable with probability function given in table.

x	-2	-1	0	1	2
$P(x_i)$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$

a) $E(x)$

b) $E(2x+3)$

c) $E(x^2)$

d) $E(2x+1)^2$

Sol:

$$\begin{aligned} \text{a) } E[X] &= \sum_{i=1}^N x_i P(x_i) \\ &= (-2)\left[\frac{1}{5}\right] + (-1)\left[\frac{2}{5}\right] + 0\left[\frac{1}{10}\right] + 1\left[\frac{1}{10}\right] + 2\left[\frac{1}{5}\right] \\ &= -\frac{2}{5} - \frac{2}{5} + \frac{1}{10} + \frac{2}{5} \\ &= \frac{-4+1}{10} \end{aligned}$$

$$E[X] = -\frac{3}{10} = -0.3$$

b) $E[2X+3]$

WKT, $E[ax+b] = aE[X] + b$
 $a=2, b=3$

$$\begin{aligned} E[2X+3] &= 2 \int_{-\infty}^{\infty} x f_X(x) dx + 3 \\ &= 2E[X] + 3 \\ &= 2(-0.3) + 3 \end{aligned}$$

$$= -0.6 + 3 = 2.4$$

(iii) $E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx$

$$= (-2)^2\left(\frac{1}{5}\right) + (-1)^2\left(\frac{2}{5}\right) + 0^2\left(\frac{1}{10}\right) + 1^2\left(\frac{1}{10}\right) + 2^2\left(\frac{1}{5}\right)$$

$$= \frac{4}{5} + \frac{2}{5} + \frac{1}{10} + \frac{1}{10} + \frac{4}{5}$$

$$= \frac{10}{5} + \frac{1}{10}$$

$$= \frac{20+1}{10} = \frac{21}{10} = 2.1$$

X	-2	-1	0	1	2
P(x)	1/5	2/5	1/10	1/10	1/5

$$(iv) E[(2x+1)^2]$$

$$E[4x^2 + 4x + 1]$$

$$= E[4x^2] + E[4x] + E[1]$$

$$= 4(2.1) + 4(-0.3) + 1$$

$$= 8.4 - 1.2 + 1$$

$$= 8.2 //$$

② If 'x' be a uniform random variable with probability density function $f_x(x) = \frac{1}{10}$, $(-5 \leq x \leq 5)$. Find $E[x]$,

a) $E[x]$

b) $E[x^2]$

c) $E[2x+5]$

d) $E[(x+1)^2]$

Sol Given, $f_x(x) = \frac{1}{10}$, $-5 \leq x \leq 5$

$$E(x) = \int_{-\infty}^{\infty} x f_x(x) dx$$

$$= \int_{-5}^5 x \left(\frac{1}{10}\right) dx$$

$$= \frac{1}{10} \left[\frac{x^2}{2} \right]_{-5}^5$$

$$= \frac{1}{20} [25 - (-25)]$$

$$= \frac{1}{20} [25 - 25]$$

$$E[x] = 0 //$$

b) $E[x^2] = \int_{-5}^5 x^2 f_x(x) dx$

$$= \frac{1}{10} \left[\frac{x^3}{3} \right]_{-5}^5$$

$$= \frac{1}{30} [125 - (-125)] = \frac{250}{30} = \frac{25}{3} = 8.33 //$$

$$\begin{aligned} \text{c) } E(2x+5) &= E(2x) + E(5) \\ &= 2E(x) + E(5) \\ &= 0 + 5 \\ &= 5 // \end{aligned}$$

$$\text{d) } E[(x+1)^2]$$

$$= E[x^2 + 1 + 2x] = E[x^2] + E[1] + E[2x]$$

$$= 8.33 + 1 + 0$$

3) consider the probability density function of Random Variable 'x' is $f_x(x) = \frac{1}{k}, [-2 \leq x \leq 3]$, then find and 0, otherwise

another Random variable $y=2x$. find the value of k,

a) k

b) $E[x]$

c) $E[y]$

d) $E[xy]$

sol: $\int_{-\infty}^{\infty} f_x(x) dx = 1$

$$\int_{-2}^3 \frac{1}{k} dx = 1$$

$$\frac{1}{k} [x]_{-2}^3 = 1$$

$$\frac{1}{k} [3 + 2] = 1$$

$$\frac{5}{k} = 1$$

$$k = 5 //$$

b) $E[x] = \int_{-\infty}^{\infty} x f_x(x) dx$

$$= \int_{-2}^3 x \left(\frac{1}{5}\right) dx$$

$$= \frac{1}{5} \left[\frac{x^2}{2} \right]_{-2}^3 = \frac{1}{5} \left[\frac{9}{2} - \frac{4}{2} \right] = \frac{1}{5} \left[\frac{5}{2} \right] = \frac{1}{2} //$$

$$= \frac{3}{\frac{1}{5}} \times dx$$

$$= \frac{1}{5} \left[\frac{x^3}{3} \right]_{-2}^3$$

$$= \frac{1}{5} [9 - (-8)]$$

$$= \frac{1}{5} (17)$$

$$= \frac{1}{5} \times 17$$

$$E(x) = 3.4$$

$$c) E(y) = E(2x)$$

$$= 2E(x)$$

$$= 2(3.4)$$

$$= 6.8$$

$$d) E(xy) = E(x) \cdot E(y)$$

$$= 3.4 \cdot 1$$

$$= 3.4$$

$$3) 1.4 \times 4.6$$

$$\frac{1.2}{2.6}$$

$$1.4$$

Moments :-

- 1) Moment about origin (mm)
- 2) Moment about mean or central mean (μ_n)

(i) Moment about origin : [mm]

Let $g(x)$ is a real valued function of Random variable 'x' such that $g(x) = x^n$, $n=0, 1, 2, \dots$ then the expectation of $g(x)$ is called moment about the origin where, 'n' indicates the order of the moment.

Let $g(x)$

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f_x(x) dx$$

$$E[g(x)] = \int_{-\infty}^{\infty} x^n f_x(x) dx$$

Case-1:

$n=0$;

$$E[x^0] = \int_{-\infty}^{\infty} x^0 f_x(x) dx$$

$$= \int_{-\infty}^{\infty} x^0 f_x(x) dx$$

$$= \int_{-\infty}^{\infty} f_x(x) dx$$

$$E[x^0] = 1 = m_0$$

→ The zero order of moment always equal to 1.

Case-2: If $n=1$;

$$E[x^1] = \int_{-\infty}^{\infty} x^1 f_x(x) dx$$

$$E[x] = \int_{-\infty}^{\infty} x f_x(x) dx = m_1$$

→ The first order of moment always equal to mean value of x.

case-3: If $n=2$,

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f_x(x) dx$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f_x(x) dx = M_2$$

→ The second order of moment always equal to mean square error value of 'x'.

2) Moment about mean (M_n): [central moment]:

Let $g(x)$ be a ^{real} function of random variable 'x' such

that $g(x) = (x - \bar{x})^n$ where $n = 0, 1, 2, 3, \dots$ then

the expected value of $g(x)$ is called as central moment (or) moment about the mean.

$$M_n = E[g(x)] = E[(x - \bar{x})^n] = \int_{-\infty}^{\infty} (x - \bar{x})^n f_x(x) dx$$

case-1: If $n=0$,

$$M_n = E[(x - \bar{x})^0] = \int_{-\infty}^{\infty} f_x(x) dx$$

$$= 1$$

$$M_0 = 1$$

→ The zero order of central moment always equals to 1.

case-2: If $n=1$

$$M_1 = E[(x - \bar{x})^1] = \int_{-\infty}^{\infty} (x - \bar{x}) f_x(x) dx$$

$$= \int_{-\infty}^{\infty} x f_x(x) dx - \bar{x} \int_{-\infty}^{\infty} f_x(x) dx$$

$$= E[x] - E[\bar{x}]$$

$$M_1 = E[x] - E[\bar{x}]$$

$$M_1 = E[x] - \bar{x}$$

$$= \bar{x} - \bar{x}$$

$$M_1 = 0$$

* First central moment always equal to 0.

Variance: (σ_x^2)

The variance of density function $f_x(x)$ is defined as second central moment of Random variable 'x'.

$$\mu_2 = E[(x - \bar{x})^2] = \int_{-\infty}^{\infty} (x - \bar{x})^2 f_x(x) dx$$

1) Find the variance of uniform density function of Random Variable x .

Sol: WKT,

Uniform density function,

$$f_x(x) = \frac{1}{b-a}$$

$a < x < b$

$$E[(x - \bar{x})^2] = \int_{-\infty}^{\infty} (x - \bar{x})^2 f_x(x) dx$$

$$\bar{x} = E[x] = \int_{-\infty}^{\infty} x f_x(x) dx$$

$$= \int_a^b x \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \int_a^b x dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b$$

$$= \frac{b^2 - a^2}{2(b-a)}$$

$$= \frac{(b+a)(b-a)}{2(b-a)}$$

$$\bar{x} = E[x] = \frac{a+b}{2}$$

variance,

$$E[(x - \bar{x})^2] = \int_a^b \left(x - \frac{a+b}{2} \right)^2 \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \int_a^b \frac{(2x - a - b)^2}{4} dx$$

$$= \frac{1}{4(b-a)} \left[\frac{(2x - a - b)^3}{6} \right]_a^b = \frac{1}{4(b-a)} \left[\frac{(2b - a - b)^3}{6} - \frac{(2a - a - b)^3}{6} \right]$$

$$= \frac{1}{2(b-a)} \left[2(b^{\bar{v}} - a^{\bar{v}}) - [b-a]a - b[b-a] \right]$$

$$= \frac{1}{2(b-a)} \left[2(b^{\bar{v}} - a^{\bar{v}}) - b^2 a \right]$$

$$= \int_a^b \left[x - \left(\frac{a+b}{2} \right) \right]^{\bar{v}} \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \int_a^b \left[x^{\bar{v}} + \left(\frac{a+b}{2} \right)^{\bar{v}} - 2x \left(\frac{a+b}{2} \right) \right] dx$$

$$= \frac{1}{b-a} \left[\frac{x^3}{3} + \left(\frac{a+b}{2} \right)^{\bar{v}} x - a+b \left(\frac{x^{\bar{v}}}{\bar{v}} \right) \right]_a^b$$

$$= \frac{1}{b-a} \left[\frac{b^3 - a^3}{3} + \left(\frac{a+b}{2} \right)^{\bar{v}} [b-a] - \frac{a+b}{2} [b^{\bar{v}} - a^{\bar{v}}] \right]$$

$$= \frac{b-a}{b-a} \left[\frac{b^3 - a^3}{3} + \left(\frac{a+b}{2} \right)^{\bar{v}} [b-a] - \frac{a+b}{2} [b^{\bar{v}} - a^{\bar{v}}] \right]$$

$$= \frac{1}{b-a} \left[\frac{b^3 - a^3}{3} + [b-a] \left[\left(\frac{a+b}{2} \right)^{\bar{v}} - \frac{a+b}{2} \right] \right]$$

$$= \frac{1}{b-a} \left[\frac{b^3 - a^3}{3} - [b-a] \left[\frac{(a+b)^{\bar{v}}}{4} \right] \right]$$

$$= \frac{b^3 - a^3}{3(b-a)} - \frac{(a+b)^{\bar{v}}}{4}$$

$$= \frac{(b-a)(b^{\bar{v}} + a^{\bar{v}} + ab)}{3(b-a)} - \frac{(a+b)^{\bar{v}}}{4}$$

$$= \frac{4b^{\bar{v}} + 4a^{\bar{v}} + 4ab - 3[a^{\bar{v}} + b^{\bar{v}} + 2ab]}{12}$$

$$= \frac{4b^{\bar{v}} + 4a^{\bar{v}} + 4ab - 3a^{\bar{v}} - 3b^{\bar{v}} - 6ab}{12}$$

$$= \frac{a^{\bar{v}} + b^{\bar{v}} - 2ab}{12}$$

$$\sigma_2^{\bar{v}} = \frac{(a-b)^{\bar{v}}}{12}$$

Standard deviation: (σ_x)

$$\sigma_x = \sqrt{(x - \bar{x})^2 f_x(x) dx}$$

1) The probability density function of Random variable of x is $f_x(x) = 0.305\sqrt{x}$ ($0 < x < 3$). Find the mean value and mean square value.

Sol: Given $f_x(x) = 0.305\sqrt{x}$

$$E(x) = \int_{-\infty}^{\infty} x f_x(x) dx$$

$$= \int_0^3 x \cdot 0.305\sqrt{x} dx$$

$$= 0.305 \int_0^3 x\sqrt{x} dx$$

$$= 0.305 \left[\frac{x^{5/2}}{5/2} \right]_0^3$$

$$= 0.305 \left[\frac{2}{5} \right] \left[3^{5/2} - 0 \right]$$

$$= \frac{0.305(2)}{5} \left[3^{2.5} \right]$$

$$= 0.12 [15.58]$$

$$E(x) = 1.9$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f_x(x) dx$$

$$= \int_0^3 x^2 \cdot 0.305\sqrt{x} dx$$

$$= 0.305 \int_0^3 \frac{x^{5/2}}{5/2} dx$$

$$= 0.305 \left[\frac{x^{7/2}}{7/2} \right]_0^3$$

$$= \frac{2(0.305)}{7} \left[3^{7/2} - 0 \right]$$

$$= \frac{2(0.305)}{7} [3^{3.5}]$$

$$= 0.0871(46.076)$$

$$E[x^v] = 40 \cdot 4.0748 //$$

Skew and co-efficient of skewness:

The skew of random variable is defined as third central moment of random variable 'x', indicating with ' μ_3 '.

$$\mu_3 = E[g(x)] = E[x - \bar{x}]^3 = \int_{-\infty}^{\infty} (x - \bar{x})^3 f_x(x) dx$$

* A random variable of $f_x(x) = \begin{cases} \frac{1}{2} \cos x, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$

Find the mean of $g(x) = 4x^v$.

Sol: Given $f_x(x) = \frac{1}{2} \cos x, -\frac{\pi}{2} < x < \frac{\pi}{2}$.

$$g(x) = 4x^v$$

$$E[x] = E[g(x)] = \int_{-\infty}^{\infty} g(x) \cdot f_x(x) dx$$

$$E[x] = E[g(x)] = \int_{-\pi/2}^{\pi/2} 4x^v \left(\frac{1}{2} \cos x\right) dx$$

$$= \frac{4}{2} \int_{-\pi/2}^{\pi/2} x^v \cos x dx$$

$$= 2 \int_{-\pi/2}^{\pi/2} x^v \cos x dx$$

$$= 2 \left[x^v (\sin x) - (2x)(-\cos x) + \int 1(-\cos x) dx \right]_{-\pi/2}^{\pi/2}$$

$$= 2 \left[x^v \sin x + 2x \cos x - \sin x \right]_{-\pi/2}^{\pi/2}$$

$$= 2 \left[\frac{\pi^v}{4} \sin \frac{\pi}{2} + \pi \cos \frac{\pi}{2} - \sin \frac{\pi}{2} + \frac{\pi^v}{4} \sin \frac{\pi}{2} - \pi \cos \frac{\pi}{2} + \sin \frac{\pi}{2} \right]$$

$$= 2 \left[\frac{\pi^v}{4} + 0 - 1 + \frac{\pi^v}{4} - 0 + 1 \right]$$

$$= 2 \left[\frac{2\pi^v}{4} + 0 \right]$$

$$= \pi^v - 8 //$$

* A random variable with PDF. $f_X(x) = \frac{5}{4}(1-x^4)$, $0 < x < 1$,
 the find $E[4x+2]$, $E[X]$.

Sol: Given $f_X(x) = \frac{5}{4}(1-x^4)$ $0 < x < 1$

$$(i) E[4x+2] = 4E[X] + 2$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

$$= \int_0^1 x \cdot \frac{5}{4}(1-x^4) dx$$

$$= \frac{5}{4} \int_0^1 x(1-x^4) dx$$

$$= \frac{5}{4} \int_0^1 (x - x^5) dx$$

$$= \frac{5}{4} \left[\frac{x^2}{2} - \frac{x^6}{6} \right]_0^1$$

$$= \frac{5}{4} \left[\frac{1}{2} - \frac{1}{6} \right]$$

$$= \frac{5}{4} \left[\frac{3-1}{6} \right]$$

$$= \frac{5}{4} \left[\frac{2}{6} \right]$$

$$= \frac{5}{12}$$

$$(ii) E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx$$

$$= \int_0^1 x^2 \left(\frac{5}{4}(1-x^4) \right) dx$$

$$= \frac{5}{4} \int_0^1 (x^2 - x^6) dx$$

$$= \frac{5}{4} \left[\frac{x^3}{3} - \frac{x^7}{7} \right]_0^1$$

$$= \frac{5}{4} \left[\frac{1}{3} - \frac{1}{7} \right]$$

$$= \frac{5}{4} \left[\frac{7-3}{21} \right]$$

$$= \frac{5}{4} \left[\frac{4}{21} \right]$$

$$= \frac{5}{21}$$

* Let 'x' be a Random Variable with probabilities shown in the table.

x	-1	1	2
P(x _i)	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

Find $E[x]$ and $E[2x+1]$ and σ_x^2

Sol: (i) $E[x] = \sum_{i=0}^N x_i P(x_i)$

$$= (-1)\left(\frac{1}{6}\right) + 1\left(\frac{1}{3}\right) + 2\left(\frac{1}{2}\right)$$

$$= -\frac{1}{6} + \frac{1}{3} + \frac{2}{2}$$

$$= -\frac{1}{6} + \frac{1}{3} + 1$$

$$= \frac{-1+2+6}{6}$$

$$= \frac{7}{6} = 1.1$$

$$E[x] = 1.1$$

(ii) $E[2x+1] = 2E[x+1]$

$$= 2(1.1) + 1$$

$$= 2.2 + 1$$

$$= 3.2$$

(iii) $\sigma_x^2 = E[x^2] - [E[x]]^2$

$$= \frac{1}{6} + \frac{1}{3} + \frac{4}{2}$$

$$= \frac{1+2+12}{6}$$

$$\left[\begin{aligned} &= \frac{15}{6} \\ &= 2.5 \end{aligned} \right]$$

$$\sigma_x^2 = E[x^2] - [E[x]]^2$$

$$= 2.5 - 1.21$$

$$\sigma_x^2 = 1.29$$

Properties of variance:

* Variance of constant always zero.

$$\text{Var}(x) = E[x - \bar{x}]^2$$

$$= E[k - \bar{k}]^2$$

$$= E[k] - E[\bar{k}]$$

$$= E[k] - \bar{k}$$

$$= E[k] - E[k]$$

$$= 0 //$$

* $\text{Var}(ax) = a^2 \text{Var}(x)$

$$\text{Var}(ax) = E[ax - a\bar{x}]^2$$

$$= E[a^2(x - \bar{x})^2]$$

$$= a^2 E[(x - \bar{x})^2]$$

$$= a^2 \text{Var}(x) //$$

* The relation between the variance and moment is given by

$$\boxed{\text{Var}(x) = (m_2 - m_1^2)}$$

$$\text{Var}(x) = E[x - \bar{x}]^2$$

$$= E[x^2 + \bar{x}^2 - 2x\bar{x}]$$

$$= E[x^2] + E[\bar{x}^2] - 2\bar{x}E[x]$$

$$= E[x^2] + \bar{x}^2 - 2\bar{x}\bar{x}$$

$$= E[x^2] + \bar{x}^2 - 2\bar{x}^2$$

$$= E[x^2] - \bar{x}^2$$

$$= m_2 - [E[x]]^2$$

$$\text{Var}(x) = m_2 - m_1^2 //$$

$$\begin{cases} m_0 = E[x^0] \\ m_1 = E[x^1] \\ m_2 = E[x^2] \end{cases}$$

* If 'x' is a random variable a and b are real constants then the variance of $ax+b = a^2 \text{Var}(x)$.

$$\boxed{\text{Var}(ax+b) = a^2 \text{Var}(x)}$$

$$\text{Var}(ax+b) = E[ax+b - \overline{(ax+b)}]^2$$

$$= E[ax+b - E[ax+b]]^2$$

$$= E[ax+b - aE[x] - b]^2$$

$$= E[ax - a\bar{x}]^2$$

$$= E[a^2(x-\bar{x})^2]$$

$$= a^2 E[(x-\bar{x})^2]$$

$$= a^2 \text{Var}(x) //$$

* If x_1 and x_2 are two independent random variables then variance of $x_1 + x_2 = \text{Var}(x_1 + x_2) = \text{Var}(x_1) + \text{Var}(x_2)$.

$$= E[x_1 + x_2 - \overline{x_1 + x_2}]^2$$

$$= E[x_1 + x_2 - \bar{x}_1 - \bar{x}_2]^2$$

$$= E[(x_1 - \bar{x}_1) + (x_2 - \bar{x}_2)]^2$$

$$= E[(x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2 + 2(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)]$$

$$= E[(x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2]$$

$$= E[(x_1 - \bar{x}_1)^2] + E[(x_2 - \bar{x}_2)^2]$$

$$= \text{Var}(x_1) + \text{Var}(x_2) //$$

Chebyshev's Inequality theorem:

If 'x' is a random variable with mean μ and variance (σ^2) then for any positive number 'k' we have.

i.e., $\boxed{P(|x - \mu| \geq k\sigma) \leq \frac{1}{k^2}}$

$$P(|x - \mu| \leq k\sigma) \geq 1 - \frac{1}{k^2}$$

let x be continuous random variable,

$$\sigma^2 = E[(x - \mu)^2]$$

$$= \int_{-\infty}^{\infty} (x - \mu)^2 f_x(x) dx$$

$$\sigma^2 = \int_{-\infty}^{\mu - k\sigma} (x - \mu)^2 f_x(x) dx + \int_{\mu - k\sigma}^{\mu + k\sigma} (x - \mu)^2 f_x(x) dx + \int_{\mu + k\sigma}^{\infty} (x - \mu)^2 f_x(x) dx$$

Neglected

$$\sigma^2 \geq \int_{-\infty}^{\mu - k\sigma} (x - \mu)^2 f_x(x) dx + \int_{\mu + k\sigma}^{\infty} (x - \mu)^2 f_x(x) dx$$

$$x \leq \mu - k\sigma$$

$$x - \mu \leq -k\sigma$$

$$-(x - \mu) \geq k\sigma$$

$$(x - \mu)^2 \geq k^2 \sigma^2$$

$$x \geq \mu + k\sigma$$

$$x - \mu \geq k\sigma$$

$$(x - \mu)^2 \geq k^2 \sigma^2$$

$$[\sigma^2 \geq \int_{-\infty}^{\mu - k\sigma} (x - \mu)^2 f_x(x) dx + \int_{\mu + k\sigma}^{\infty} (x - \mu)^2 f_x(x) dx]$$

$$\sigma^2 \geq \int_{-\infty}^{\mu - k\sigma} k^2 \sigma^2 f_x(x) dx + \int_{\mu + k\sigma}^{\infty} k^2 \sigma^2 f_x(x) dx$$

$$\sigma^2 \geq k^2 \sigma^2 \left(\int_{-\infty}^{\mu - k\sigma} f_x(x) dx + \int_{\mu + k\sigma}^{\infty} f_x(x) dx \right)$$

$$\frac{1}{k^2} \geq \left(\int_{-\infty}^{\mu - k\sigma} f_x(x) dx + \int_{\mu + k\sigma}^{\infty} f_x(x) dx \right)$$

$$\frac{1}{k^2} \geq P(x \leq \mu - k\sigma) + P(\mu + k\sigma < x) \left[\int_{-\infty}^{\mu - k\sigma} f_x(x) dx = F_x(\mu - k\sigma) = P(x \leq \mu - k\sigma) \right]$$

$$\frac{1}{k^2} \geq P((x - \mu) \leq -k\sigma) + P((x - \mu) \geq k\sigma) \left(P(x_1 < x \leq x_2) = \int_{x_1}^{x_2} f_x(x) dx \right)$$

$$\frac{1}{k^2} \geq P(-(x - \mu) \geq k\sigma) + P(x - \mu \geq k\sigma)$$

$$\frac{1}{k^2} \geq P(|x - \mu| \geq k\sigma)$$

$$P(|x - \mu| \leq k\sigma) \leq \frac{1}{k^2}$$

* calculate Expected value of $E[x^3]$ for the uniform probability density function.

Sol: $f_x(x) = \frac{1}{b-a}, a < x < b$

$$\begin{aligned} E[x] &= \int_{-\infty}^{\infty} x^3 f_x(x) dx \\ &= \int_a^b x^3 \frac{1}{b-a} dx \\ &= \frac{1}{b-a} \int_a^b x^3 dx \\ &= \frac{1}{b-a} \left[\frac{x^4}{4} \right]_a^b \\ &= \frac{1}{b-a} \left(\frac{b^4}{4} - \frac{a^4}{4} \right) \end{aligned}$$

$$\begin{aligned} &= \frac{(b^4 - a^4)}{4(b-a)} \\ &= \frac{(b^3 - a^3)(b + a)}{4(b-a)} \\ &= \frac{(b-a)(b^2 + ab + a^2)(b+a)}{4(b-a)} \\ &= \frac{(a+b)(a^2 + ab + b^2)}{4} \end{aligned}$$

Characteristic function: $(\phi_x(\omega))$:

Consider a Random Variable $x(\omega)$ with its probability density function $f_x(x)$, then the expected value of function is $E[e^{j\omega x}]$ is called characteristic function.

$$\phi_x(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} e^{j\omega x} f_x(x) dx$$

Theorem: If $\phi_x(w)$ is a characteristic function of x

-function of 'x' is $\phi_x(\omega) = E[e^{j\omega x}]$

WKT,

$$e^{j\omega x} = 1 + j\omega x + \frac{j^2 \omega^2 x^2}{2!} + \frac{j^3 \omega^3 x^3}{3!} + \dots + \frac{j^n \omega^n x^n}{n!}$$

$$e^{j\omega x} = 1 + j\omega x - \frac{\omega^2 x^2}{2!} - \frac{j\omega^3 x^3}{6} + \dots + \frac{j^n \omega^n x^n}{n!}$$

$$E[e^{j\omega x}] = E[1] + E[j\omega x] - E\left[\frac{\omega^2 x^2}{2}\right] - E\left[\frac{j\omega^3 x^3}{6}\right]$$

$$= E[1] + j\omega E[x] - \frac{\omega^2}{2} E[x^2] - \frac{j\omega^3}{6} E[x^3] + \dots$$

$$\phi_X(\omega) = m_0 + j\omega m_1 - \frac{\omega^2}{2} m_2 - \frac{j\omega^3}{6} m_3 + \dots$$

$$\left. \frac{d}{d\omega} (\phi_x(\omega)) \right|_{\omega=0} = 0 + jM_1 - \omega M_2 - \frac{3j\omega^2}{6} M_3 + \dots \rightarrow (1)$$

$$\frac{\delta}{\delta \omega} (\phi_x(\omega)) = \hat{J}^M, \quad \frac{(\hat{v}^{\mu\nu}) (\hat{v}^\mu - \hat{v}^\nu)}{(\hat{v}^{\mu\nu}) (\hat{v}^\mu - \hat{v}^\nu)} =$$

$$m_1 = \frac{1}{j} \frac{d}{d\omega} (\phi_x(\omega))$$

$$M_1 = (-j) \frac{\partial}{\partial \omega} (\phi_X(\omega))$$

Diff ① w.r.to 'w', then

$$\frac{\delta^v}{\delta \omega} (\phi_x(\omega)) = 0 + 0 - \dot{m}_2$$

$$\frac{\delta^v}{\delta \omega^v} (\phi_X(\omega)) = -m_2$$

$$m_2 = \left(\frac{1}{j} \right) \frac{\partial}{\partial \omega} (\phi_x(\omega))$$

$$M_2 = -j^\nu \frac{d^\nu}{d\omega^\nu} (\phi_k(\omega))$$

$$m_n = (-j)^n \frac{d^n}{d\omega^n} (\phi_x(\omega))$$

properties of characteristic function:

1) If $\phi_x(\omega)$ is a characteristic function then the function is '1' at $\omega = 0$.

$$\begin{aligned} E[e^{j\omega x}] &= \int_{-\infty}^{\infty} e^{j\omega x} f_x(x) dx \\ &= \int_{-\infty}^{\infty} e^0 f_x(x) dx \\ &= \int_{-\infty}^{\infty} f_x(x) dx \\ &= 1 \end{aligned}$$

$$\phi_x(\omega) = 1$$

$$\boxed{\phi_x(0) = 1}$$

2) The maximum amplitude of characteristic function is unity at $\omega = 0$.

$$|\phi_x(\omega)| \leq 1$$

$$\begin{aligned} |\phi_x(\omega)| &= |E[e^{j\omega x}]| \leq \int_{-\infty}^{\infty} |e^{j\omega x} f_x(x)| dx \\ &\leq \int_{-\infty}^{\infty} |e^{j\omega x}| |f_x(x)| dx \\ &\leq \int_{-\infty}^{\infty} |f_x(x)| dx \\ &\leq 1 \end{aligned}$$

3) If $\phi_x(\omega)$ is a characteristic function, $-\infty < \omega < \infty$.

4) If $\phi_x(-\omega)$ and $\phi_x(\omega)$ are conjugate functions then,

$$(i) \phi_x(-\omega) = \phi_x^*(\omega)$$

$$(ii) \phi_x^*(-\omega) = \phi_x(\omega)$$

③ $\phi_x(\omega)$ is a continuous function in range $-\infty < \omega < \infty$

$$\phi_x(\omega) = E[e^{j\omega x}]$$

$$\phi_x(-\omega) = E[e^{-j\omega x}]$$

$$= E[e^{j\omega x}]^*$$

$$= \phi_x^*(\omega)$$

$$\boxed{\phi_x^*(-\omega) = \phi_x(\omega)}$$

$$\phi_x(\omega) = E[e^{j\omega x}]$$

$$\phi_x^*(\omega) = E[e^{j\omega x}]^*$$

$$\phi_x^*(-\omega) = E[e^{-j\omega x}]^*$$

$$= E[e^{j\omega x}]$$

$$= \phi_x(\omega)$$

$$1 = (\omega)_x \phi$$

$$\boxed{1 = (\omega)_x \phi}$$

$$\boxed{\phi_x^*(-\omega) = \phi_x(\omega)}$$

* If $\phi_x(\omega)$ is characteristic function of Random variable x , then another Random variable $y = ax + b$, where a, b are real constants, then $\phi_y(\omega) = e^{j\omega b} \phi_x(a\omega)$

$$\phi_y(\omega) = E[e^{j\omega y}]$$

$$= E[e^{j\omega(ax+b)}]$$

$$= E[e^{j\omega ax} * e^{j\omega b}]$$

$$= e^{j\omega b} E[e^{j\omega ax}]$$

$$= e^{j\omega b} \phi_x(a\omega)$$

$$\boxed{\phi_y(\omega) = e^{j\omega b} \phi_x(a\omega)}$$

$$(\omega)_x \phi = (\omega)_x \phi$$

$$(\omega)_x \phi = (\omega)_x \phi$$

$$(\omega)_x \phi = (\omega)_x \phi$$

* If $\phi_x(\omega)$ is characteristic function of random variable 'x' then $\phi_x(c\omega) = \phi_{cx}(\omega)$.

Sol:

$$\begin{aligned}\phi_x(c\omega) &= E[e^{jc\omega x}] \\ &= E[e^{jcx\omega}] \\ &= \phi_{cx}(\omega)\end{aligned}$$

$$\boxed{\phi_x(c\omega) = \phi_{cx}(\omega)}$$

* If x_1 and x_2 are two independent random variables the $\phi_{x_1+x_2}(\omega) = \phi_{x_1}(\omega) \cdot \phi_{x_2}(\omega)$

$$\begin{aligned}\phi_{x_1+x_2}(\omega) &= E[e^{j(x_1+x_2)\omega}] \\ &= E[e^{jx_1\omega} \cdot e^{jx_2\omega}] \\ &= E[e^{j\omega x_1}] \cdot E[e^{j\omega x_2}] \\ &= \phi_{x_1}(\omega) \cdot \phi_{x_2}(\omega)\end{aligned}$$

$$\boxed{\phi_{x_1+x_2}(\omega) = \phi_{x_1}(\omega) \cdot \phi_{x_2}(\omega)}$$

Moment generating function:

If $M_x(v)$ is moment generating function of random variable 'x' then n th moment of Random variable 'x' is given by the expected value of function e^{vx} . ($E[e^{vx}]$)

$$\boxed{M_x(v) = E[e^{vx}] = \int_{-\infty}^{\infty} e^{vx} f_x(x) dx}$$

Theorem: If $M_x(v)$ is moment generating function of random variable 'x' then n th moment of function is given by $m_n = \frac{d^n}{dv^n} M_x(v) \Big|_{v=0}$

Proof:

$$M_X(v) = E[e^{vx}]$$

$$e^{vx} = 1 + vx + \frac{v^2 x^2}{2!} + \frac{v^3 x^3}{3!} + \dots$$

$$E[e^{vx}] = E[1] + E[vx] + E\left[\frac{v^2 x^2}{2}\right] + \dots$$

$$= E[1] + vE[x] + \frac{v^2}{2}E[x^2] + \dots$$

$$M_X(v) = M_0 + vM_1 + \frac{v^2}{2}M_2 + \dots \quad \text{①}$$

$$\frac{\partial}{\partial v} M_X(v) = 0 + M_1 + \frac{2v}{2}M_2 + \dots \quad \text{②}$$

$$\boxed{\frac{\partial}{\partial v} M_X(v) = M_1}$$

diff ② w.r.to 'v'

$$\frac{\partial^2}{\partial v^2} M_X(v) = 0 + 0 + M_2 + \dots$$

$$\boxed{\frac{\partial^2}{\partial v^2} M_X(v) = M_2}$$

$$\text{Hence, } \boxed{m_n = \frac{\partial^n}{\partial v^n} M_X(v)}$$

Properties:

(1) The moment generating function of random variable at $v=0$ is unity.

$$M_X(v) = E[e^{vx}]$$

$$E[e^{vx}] = \int_{-\infty}^{\infty} e^{vx} f_X(x) dx$$

$$= \int_{-\infty}^{\infty} e^{0x} f_X(x) dx$$

$$= 1$$

2) Let 'x' be a random variable with its moment generating function $M_x(v)$, and another random variable $Y = ax + b$ is given by $M_Y(v) = e^{bv} \cdot M_x(av)$

$$M_x(v) = E[e^{vx}]$$

$$M_Y(v) = E[e^{vY}]$$

$$= E[e^{v(ax+b)}]$$

$$= (E[e^{avx}]) \cdot (E[e^{vb}])$$

$$M_Y(v) = (M_x(av)) \cdot (e^{vb})$$

$$M_Y(v) = M_x(av) \cdot e^{bv}$$

3) If $M_x(v)$ is the moment generating function of random variable 'x' then $M_x(cv) = M_x(v)$.

$$M_x(cv) = E[e^{cvx}]$$

$$= E[e^{cxv}]$$

$$= M_{cx}(v)$$

$$M_x(cv) = M_{cx}(v) //$$

$$4) M_{x_1+x_2}(v) = M_{x_1}(v) \cdot M_{x_2}(v).$$

$$M_{x_1+x_2}(v) = E[e^{(x_1+x_2)v}]$$

$$= E[e^{x_1v} \cdot e^{x_2v}]$$

$$= E[e^{x_1v}] \cdot E[e^{x_2v}]$$

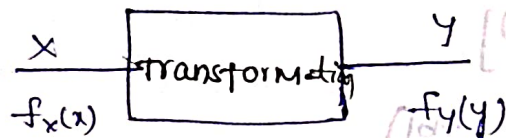
$$= M_{x_1}(v) \cdot M_{x_2}(v)$$

$$M_{x_1+x_2}(v) = M_{x_1}(v) \cdot M_{x_2}(v).$$

Transformation of a Random Variable:

It is used to convert one random variable 'x' into another random variable 'y'. The function may be linear or non-linear.

$$Y = T[X]$$



→ In a transformation of Random Variable, we need to consider 3 cases. They are

(i) Both x and T are continuous and the function is either monotonically increasing (a) decreasing.

(ii) Both x and T are continuous and the function is not monotonic.

(iii) x is discrete and T is continuous function

→ consider a Random Variable 'x' the transformation of two random variables is given by

$$T(x_1) < T(x_2) \quad \text{for } x_1 < x_2$$

then the function is monotonically increasing function

→ consider a Random Variable 'x' the transformation of two random variables is given by

$$T(x_2) < T(x_1) \quad \text{for } x_2 < x_1$$

then the function is monotonically decreasing function

→ find the variance of poisson distribution function,

Monotonic Increasing function:

Let 'T' be the continuous and differentiable at x_0 ,
 $f_x(x) \neq 0$, if

y_0 is any particular value of y , the relation between

x_0 and y_0 is given by

$$y_0 = T(x_0)$$

$$x_0 = T^{-1}(y_0)$$

let x and y be the probability events of $y \leq y_0$,
 $X \leq x_0$ be equal so,

$$Y \leq y_0 = X \leq x_0$$

$$P(Y \leq y_0) = P(X \leq x_0)$$

$$F_Y(y_0) = F_X(x_0)$$

$$\int_{-\infty}^{y_0} f_Y(y) dy = \int_{-\infty}^{x_0} f_X(x) dx$$

—Applying derivative on both sides

$$\frac{d}{dy_0} \int_{-\infty}^{y_0} f_Y(y) dy = \frac{d}{dx_0} \int_{-\infty}^{x_0} f_X(x) dx$$

$$f_Y(y_0) = \frac{dx_0}{dy_0} f_X(x_0)$$

$$f_Y(y_0) = \frac{d(T^{-1}(y_0))}{dy_0} f_X(T^{-1}(y_0))$$

$$x = x_0, y = y_0$$

$$f_Y(y) = \frac{d}{dy} T^{-1}(y) f_X(T^{-1}(y))$$

$$f_Y(y) = \frac{dx}{dy} f_X(x)$$

UNIT-2:

1) Find the variance of Gaussian density function.

Sol:

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\mu)^2}{2\sigma_x^2}}$$

$$E(x) = \int_{-\infty}^{\infty} x f_x(x) dx$$

$$= \int_{-\infty}^{\infty} x \left[\frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\mu)^2}{2\sigma_x^2}} \right] dx$$

let $\frac{x-\mu}{\sigma_x} = y$, $\begin{cases} x-\mu = \sigma_x y \\ x = \sigma_x y + \mu \end{cases}$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{y^2}{2}} dx = \sigma_x dy$$

$$= \int_{-\infty}^{\infty} (\sigma_x y + \mu) \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{y^2}{2}} (\sigma_x) dy$$

$$= \int_{-\infty}^{\infty} (\sigma_x y + \mu) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

$$= \int_{-\infty}^{\infty} \frac{\sigma_x y + \mu}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma_x y e^{-\frac{y^2}{2}} dy + \mu \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma_x y e^{-\frac{y^2}{2}} dy + \int_{-\infty}^{\infty} \mu e^{-\frac{y^2}{2}} dy$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 + \int_{-\infty}^{\infty} \mu e^{-y^2/2} dy \right]$$

$$= \frac{1}{\sqrt{2\pi}} (\mu \sqrt{2\pi})$$

$$\boxed{E(x) = \mu}$$

$$\text{variance} = E[x - \bar{x}]^2$$

$$= E[x - \mu]^2$$

$$E[x - \mu]^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f_x(x) dx$$

$$= \int_{-\infty}^{\infty} x^2 f_x(x) dx + \int_{-\infty}^{\infty} \mu^2 f_x(x) dx - \int_{-\infty}^{\infty} 2\mu x f_x(x) dx$$

$$[a]$$

$$\sigma_x^2 = E[x^2] - E[x]^2$$

$$\sigma_x^2 = E[x^2] - E[x]^2$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f_x(x) dx$$

$$= \int_{-\infty}^{\infty} x^2 \left[\frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-\mu)^2}{2\sigma_x^2}} \right] dx$$

$$\text{let } \frac{x-\mu}{\sigma_x} = y$$

$$x - \mu = \sigma_x y$$

$$x = \sigma_x y + \mu$$

$$dx = \sigma_x dy$$

$$= \frac{1}{\sqrt{2\pi}\sigma_x} \int_{-\infty}^{\infty} (\sigma_x y + \mu)^2 e^{-y^2/2} dy \cdot \sigma_x$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma_x y + \mu) e^{-y^2/2} dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma_x y e^{-y^2/2} dy + \int_{-\infty}^{\infty} \mu e^{-y^2/2} dy + 2 \int_{-\infty}^{\infty} \mu y e^{-y^2/2} dy$$

$$= \frac{1}{\sqrt{2\pi}} \left[\sigma_x \sqrt{2\pi} + \mu \sqrt{2\pi} + 0 \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\sigma_x \sqrt{2\pi} + \mu \sqrt{2\pi} + 0 \right] = \frac{1}{\sqrt{2\pi}} \left[\sqrt{2\pi} (\mu + \sigma_x) \right]$$

$$E[x] = \mu$$

$$E[x] = \mu + \sigma_x$$

$$\therefore \sigma_x = E[x] - E[x]$$

$$= \mu + \sigma_x - \mu$$

$$= \sigma_x$$

$$\sigma_x = \mu + \sigma_x - \mu$$

$$\sigma_x = \sigma_x$$

$$\sigma_x = \sigma_x$$

UNIT-II
continuation:

Monotonic decreasing function:

Let us consider,

$$F_y(y_0) = 1 - F_x(x_0)$$

$$\int_{-\infty}^{y_0} f_y(y) dy = 1 - \int_{-\infty}^{x_0} f_x(x) dx$$

$$\frac{d}{dy} \int_{-\infty}^{y_0} f_y(y) dy = 0 - \frac{d}{dy} \int_{-\infty}^{x_0} f_x(x) dx$$

$$f_y(y_0) = - \frac{dx_0}{dy} f_x(x_0)$$

$$= - f_x(x_0) \frac{dx_0}{dy}$$

Replace, $x_0 = x$, $y_0 = y$

$$\therefore -f_y(y) = -\frac{dx}{dy} f_x(x)$$

mean and

* Find the variance of poisson distribution function.

Sol:

$$E[x] = \int_{-\infty}^{\infty} x f_x(x) dx$$

$$= \sum_{n=0}^{\infty} n \cdot p(n)$$

$$p(n) = \frac{e^{-\lambda} \lambda^n}{n!}$$

$$= \sum_{n=0}^{\infty} n \cdot \frac{e^{-\lambda} \lambda^n}{n!}$$

$$\text{If } \lambda = 0$$

$$p(n) = 0 + \frac{e^{-\lambda} \lambda}{1} + \frac{e^{-\lambda} \lambda^2}{2!} + \frac{e^{-\lambda} \lambda^3}{3!} + \dots$$

$$= e^{-\lambda} \lambda \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots \right]$$

$$= e^{-\lambda} \cdot \lambda [e^{\lambda}]$$

$$= \lambda [e^{-\lambda} + \lambda]$$

$$= \lambda [e^0]$$

$$\boxed{E[x] = \lambda}$$

$$\text{Variance, } \sigma_x^2 = E[x^2] - (E[x])^2$$

$$E[x^2] = \sum_{n=0}^{\infty} n^2 p(n)$$

$$= \sum_{n=0}^{\infty} n^2 \cdot \frac{e^{-\lambda} \lambda^n}{n!}$$

$$[\because x^2 = x + x(x-1)]$$

$$= \sum_{n=0}^{\infty} x + x(x-1) \cdot \frac{e^{-\lambda} \lambda^n}{n!}$$

$$= \sum_{x=0}^{\infty} \frac{x e^{-\lambda} \lambda^x}{x!} + \sum_{x=0}^{\infty} \frac{x(x-1) e^{-\lambda} \lambda^x}{x!}$$

$$= \lambda + \sum_{x=0}^{\infty} \frac{x(x-1) e^{-\lambda} \lambda^x}{x!}$$

$$= \lambda + 0 + 0 + \frac{2 e^{-\lambda} \lambda^2}{2 \times 1} + \frac{(3 \times 2) e^{-\lambda} \lambda^3}{3 \times 2} + \frac{4(3) e^{-\lambda} \lambda^4}{4 \times 3 \times 2 \times 1} + \dots$$

$$= \lambda + e^{-\lambda} \lambda^2 + e^{-\lambda} \lambda^3 + \frac{e^{-\lambda} \lambda^4}{2!} + \dots$$

$$= \lambda + e^{-\lambda} \lambda^2 \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \dots \right]$$

$$= \lambda + e^{-\lambda} \lambda^2 (e^{\lambda})$$

$$= \lambda + \lambda^2 (1)$$

$$E(x) = \lambda(\lambda + 1)$$

$$\therefore \sigma_x^2 = E(x^2) - E(x)^2$$

$$= \lambda + \lambda^2 - \lambda^2 = \lambda$$

$$\therefore \sigma_x^2 = \lambda$$

$$[h_0] h_0 =$$

$$[h_0 + h_1] h_0 =$$

$$[h_0] h_1 =$$

$$[h_0] h_2 =$$

$$Variance: E(x^2) - (E(x))^2$$

$$E(x) = \sum_{x=0}^{\infty} x p(x)$$

$$\sum_{x=0}^{\infty} \frac{x e^{-\lambda} \lambda^x}{x!}$$

$$\sum_{x=0}^{\infty} \frac{x e^{-\lambda} \lambda^x}{x!} = \lambda$$

UNIT-III MULTIPLE RANDOM VARIABLES

probability Distribution function:

let two events $A = \{X \leq x\}$, $B = \{Y \leq y\}$ then the probability distribution function can be defined as

$$F_{xy}(x, y) = P(X \leq x, Y \leq y)$$

$$F_{xy}(x, y) = P(X \leq x \cap Y \leq y)$$

probability distribution in terms of discrete Random variable:

For discrete Random variables let $x = \{x_1, x_2, \dots, x_n\}$ and $y = \{y_1, y_2, \dots, y_n\}$. The joint CDF of x and y is defined as

$$F_{xy}(x, y) = \sum_{m=1}^M \sum_{n=1}^N P(x_n, y_m) u(x - x_m) u(y - y_n)$$

properties of PDF:

$$(1) F_{xy}(-\infty, \infty) = 0$$

$$F_{xy}(-\infty, -\infty) = 0$$

$$P_{xy}(x, y) = P(X \leq x, Y \leq y)$$

$$= P(X \leq -\infty, Y \leq -\infty)$$

$$= P(X \leq -\infty \cap Y \leq -\infty)$$

$$= P(\phi \cap \phi)$$

$$= P(\phi)$$

$$F_{xy}(-\infty, -\infty) = 0$$

$$(2) F_{xy}(x, -\infty) = 0$$

$$P(X \leq x, Y \leq y) = P(X \leq x \cap Y \leq y)$$

$$= P(X \leq x \cap \phi)$$

$$= P(\{x_1, x_2, x_3, \dots, x_n\} \cap \{0\})$$

$$= P(\phi)$$

$$F_{xy}(x, -\infty) = 0$$

$$(3) F_{xy}(-\infty, y) = 0$$

$$\begin{aligned} P(X \leq x, Y \leq y) &= P(X \leq -\infty \cap Y \leq y) \\ &= P(X \leq \phi \cap Y \leq y) \\ &= P(\{0\} \cap \{y, 2, 3, \dots, y\}) \\ &= P(\phi) \end{aligned}$$

$$F_{xy}(x, -\infty) = 0$$

$$(4) F_{xy}(\infty, \infty) = 1$$

$$\begin{aligned} P(X \leq x, Y \leq y) &= P(X \leq \infty \cap Y \leq \infty) \\ &= P(S \cap S) \\ &= P(S) \end{aligned}$$

$$F_{xy}(\infty, \infty) = 1$$

(5) Marginal Distribution function:

$$1) F_{xy}(x, \infty) = F_x(x)$$

$$\begin{aligned} F_{xy}(x, \infty) &= P(X \leq x \cap Y \leq \infty) \\ &= P(X \leq x \cap S) \\ &= P(X \leq x) \end{aligned}$$

$$F_{xy}(x, \infty) = F_x(x)$$

$$6) F_{xy}(\infty, y) = F_y(y)$$

$$\begin{aligned} F_{xy}(\infty, y) &= P(X \leq \infty \cap Y \leq y) \\ &= P(S \cap Y \leq y) \\ &= P(Y \leq y) \end{aligned}$$

$$F_{xy}(\infty, y) = F_y(y)$$

7) The distribution function bounded between 0 and 1.

$$F_{xy}(-\infty, -\infty) < F_{xy}(x, y) < F_{xy}(\infty, \infty)$$

$$F_{xy}(-\infty, -\infty) = 0$$

$$F_{xy}(\infty, \infty) = 1$$

$$0 < F_{xy}(x, y) < 1$$

8) probability Density function (Joint Density Function):

The second order derivative of distribution function is called as joint probability density function.

$$\therefore \frac{\partial^2}{\partial x \partial y} [F_{xy}(x, y)] = f_{xy}(x, y)$$

properties of probability density function:

1) $f_{xy}(x, y) \geq 0$

WKT, the distribution function lies in between 0 and 1. So, the derivative of distribution function also lies in between 0 and 1. [Density function is non-negative].

2) $\int_{-\infty}^{\infty} f_{xy}(x, y) dx = 1$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{xy}(x, y) dx dy = 1$$

$$\rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial^2}{\partial x \partial y} [F_{xy}(x, y)] dx dy$$

$$= \int_{-\infty}^{\infty} \frac{d}{dy} \left[\int_{-\infty}^{\infty} \frac{d}{dx} [F_{xy}(x, y)] dx \right] dy$$

$$= \int_{-\infty}^{\infty} \frac{d}{dy} [F_{xy}(x, y)]_{-\infty}^{\infty} dy$$

$$= \int_{-\infty}^{\infty} \frac{d}{dy} [F_{xy}(\infty, y) - F_{xy}(-\infty, y)] dy$$

$$= \int_{-\infty}^{\infty} \frac{d}{dy} [F_y(y)] - [0] dy$$

$$= \int_{-\infty}^{\infty} \frac{d}{dy} [F_y(y)] dy$$

$$= [F_y(y)]_{-\infty}^{\infty}$$

$$= F_y(\infty) - F_y(-\infty)$$

$$= 1$$

$$3) F_{xy}(x, y) = \int_{-\infty}^y \int_{-\infty}^x F_{xy}(x, y) dx dy$$

$$F_{xy}(x, y) = \int_{-\infty}^y \int_{-\infty}^x F_{xy}(x, y) dx dy$$

$$= \int_{-\infty}^y \int_{-\infty}^x \frac{\partial}{\partial x} [F_{xy}(x, y)] dx dy$$

$$= \int_{-\infty}^y \frac{d}{dy} \left[\int_{-\infty}^x \frac{d}{dx} [F_{xy}(x, y)] dx \right] dy$$

$$= \int_{-\infty}^y \frac{d}{dy} [F_{xy}(x, y)]_{-\infty}^x dy$$

$$= \int_{-\infty}^y \frac{d}{dy} [F_{xy}(x, y) - F_{xy}(-\infty, y)] dy$$

$$= \int_{-\infty}^y \frac{d}{dy} [F_{xy}(x, y) - 0] dy$$

$$= \int_{-\infty}^y \frac{d}{dy} [F_{xy}(x, y)] dy$$

$$= [F_{xy}(x, y)]_{-\infty}^y$$

$$= [F_{xy}(x, y) - F_{xy}(x, -\infty)]$$

$$= F_{xy}(x, y) - 0$$

$$= F_{xy}(x, y) //$$

$$4) \int_{y_1}^{y_2} \int_{x_1}^{x_2} f_{xy}(x, y) dx dy = f(x_1, y_1) - f(x_2, y_1) + f(x_2, y_2) - f(x_1, y_2)$$

$$\int_{y_1}^{y_2} \int_{x_1}^{x_2} f_{xy}(x, y) dx dy = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{\partial}{\partial x} [F_{xy}(x, y)] dx dy$$

$$= \int_{y_1}^{y_2} \frac{d}{dy} \left[\int_{x_1}^{x_2} \frac{\partial}{\partial x} [F_{xy}(x, y)] dx \right] dy$$

$$= \int_{y_1}^{y_2} \frac{d}{dy} [F_{xy}(x, y)]_{x_1}^{x_2} dy$$

$$= \int_{y_1}^{y_2} \frac{d}{dy} [F_{xy}(x_2, y) - F_{xy}(x_1, y)] dy$$

$$\begin{aligned}
 &= \left[F_{xy}(x_2, y) - F_{xy}(x_1, y) \right]_{y_1}^{y_2} \\
 &= F_{xy}(x_2, y_2) - F_{xy}(x_1, y_2) - F_{xy}(x_2, y_1) + F_{xy}(x_1, y_1) \\
 &= P(x_1 < X < x_2, y_1 < Y < y_2)
 \end{aligned}$$

* Expected value of $E[X+Y] = E[X] + E[Y]$

sol:

$$E[X+Y] = E[X] + E[Y]$$

$$\begin{aligned}
 E[X+Y] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x+y) f_x(x) f_y(y) dx dy \\
 &= \int_{-\infty}^{\infty} x f_x(x) dx + \int_{-\infty}^{\infty} y f_y(y) dy \cdot f_x(x) dx \\
 &= E[X] + E[Y] \quad \left\{ \because \int_{-\infty}^{\infty} f_y(y) dy = 1 \right. \\
 &\quad \left. \int_{-\infty}^{\infty} f_x(x) dx = 1 \right\}
 \end{aligned}$$

$$\therefore E[X+Y] = E[X] + E[Y]$$

Marginal density function:

$$\frac{d}{dx} [F_x(x)] = f_x(x) = F_x(x, \infty) - (F_x(x))_{x=\infty}$$

The derivative of marginal distribution function is called marginal density function.

$$\boxed{\frac{d}{dx} [F_x(x)] = \int_{-\infty}^{\infty} f_{xy}(x, y) dy}$$

$$\text{Ily, } \boxed{\frac{d}{dy} [F_y(y)] = \int_{-\infty}^{\infty} f_{xy}(x, y) dx}$$

conditional joint distribution and density functions:

→ If A and B are two events, where $A = x \leq x$ and $B = y \leq y$ then the ^{joint} conditional distribution function is

$$F_{x/y}(x/y) = \frac{P(x \leq x, y \leq y)}{P(y \leq y)}$$

$$= \frac{P(x \leq x \cap y \leq y)}{P(y \leq y)}$$

$$= \frac{F_{xy}(x, y)}{F_y(y)}$$

$$F_{x,y}(x, y)$$

$$f_{x,y}(x, y)$$

$$\frac{d}{dx} [F_x(x)]$$

$$f_x(x)$$

$$F_x(x), \int f_x(x)$$

point

condition:

* The conditional joint distribution of random variable 'x' is given that 'y' at a specific value i.e.

$y = y$ defined as,

$$F_{x/y}(x/y = y) = \frac{\int_{-\infty}^x f_{x,y}(x, y) dx}{f_y(y)}$$

* The conditional joint density of random variable 'x' is given that 'y' at a specific value i.e., $y = y$ defined as

$$f_{x/y}(x/y = y) = \frac{f_{xy}(x, y)}{f_y(y)}$$

proof:

let distribution function of Random variable 'x' given that event 'y' is defined as.

$$F_{x/y}(x/y) = \frac{P(x \leq x, y \leq y)}{P(y \leq y)}$$

$$= P(X \leq x \cap Y \leq y)$$

$$P(Y \leq y)$$

$$= P(X \leq x \cap Y)$$

$$P(Y)$$

$$[\because Y = y]$$

Let event 'y' for point condition.

$$y = y - \Delta y < y \leq y + \Delta y$$

$$F_{x/y}(x/y) = \frac{P(X \leq x \cap y - \Delta y < y \leq y + \Delta y)}{P(y - \Delta y < y \leq y + \Delta y)}$$

$$P(y - \Delta y < y \leq y + \Delta y)$$

$$\int_{y-\Delta y}^{y+\Delta y} \int_{-\infty}^{\infty} f_{xy}(x,y) dx dy$$

$$= \frac{\int_{y-\Delta y}^{y+\Delta y} \int_{-\infty}^{\infty} f_x(x) f_y(y) dx dy}{\int_{y-\Delta y}^{y+\Delta y} f_y(y) dy}$$

$$= \frac{\int_{-\infty}^{\infty} f_x(x) dx \int_{y-\Delta y}^{y+\Delta y} f_y(y) dy}{\int_{y-\Delta y}^{y+\Delta y} f_y(y) dy}$$

$$= \int_{-\infty}^{\infty} f_x(x) dx$$

$$= F_x(x)$$

$$F_{x/y}(x/y) = \frac{\int_{-\infty}^x \int_{y-\Delta y}^{y+\Delta y} f_{xy}(x,y) dx dy}{\int_{y-\Delta y}^{y+\Delta y} f_y(y) dy}$$

$$F_{x/y}(x/y) = \frac{\int_{-\infty}^x f_x(x) dx}{f_y(y)}$$

$$F_{x/y}(y=y) = \frac{f_y(y)}{f_y(y)}$$

$$F_{x/y}(y=y) = \frac{\int_{-\infty}^x f_{xy}(x,y) dx}{f_y(y)}$$

Apply ' $\frac{d}{dx}$ ' on both sides.

$$\frac{d}{dx} [F_{x/y}(y=y)] = \frac{d}{dx} \left[\frac{\int_{-\infty}^x f_{xy}(x,y) dx}{f_y(y)} \right]$$

$$f_{x/y}(x/y) = \frac{f_{xy}(x,y)}{f_y(y)}$$

Interval condition:

The conditional distribution function of random variable 'x' given that 'y' is in a specified interval $y_1 < y < y_2$ is defined as:

$$F_{x/y}(x/y) = \frac{\int_{y_1}^{y_2} \int_{-\infty}^x f_{xy}(x,y) dx dy}{\int_{y_1}^{y_2} f_y(y) dy}$$

$$= P(x \leq x, y_1 < y < y_2)$$

$$P(y_1 < y < y_2)$$

$$= \frac{\int_{y_1}^{y_2} \int_{-\infty}^x f_{xy}(x,y) dx dy}{\int_{y_1}^{y_2} f_y(y) dy}$$

$$= \int_{y_1}^{y_2} \int_{-\infty}^{\infty} \frac{d}{dx} [F_{xy}(x, y)] dx dy$$

$$\frac{\int_{y_1}^{y_2} \frac{d}{dy} [F_y(y)] dy}{\int_{y_1}^{y_2} \frac{d}{dy} [F_y(y)] dy}$$

$$= \int_{y_1}^{y_2} \frac{d}{dy} [F_{xy}(x, y)] dy$$

$$\frac{\int_{y_1}^{y_2} \frac{d}{dy} [F_{xy}(x, y)] dy}{\int_{y_1}^{y_2} \frac{d}{dy} [F_y(y)] dy}$$

$$= \int_{y_1}^{y_2} [F_{xy}(x, y_2) - F_{xy}(x, y_1)] dy$$

$$\frac{[F_y(y)]_{y_1}^{y_2}}{[F_y(y)]_{y_1}^{y_2}} = (F_y(y))_{y_1}^{y_2}$$

$$= \int_{y_1}^{y_2} \frac{d}{dy} [F_{xy}(x, y)] dy$$

$$= \frac{F_y(y_2) - F_y(y_1)}{F_y(y_2) - F_y(y_1)}$$

$$= \frac{F_{xy}(x, y_2) - F_{xy}(x, y_1)}{F_y(y_2) - F_y(y_1)}$$

$$= \frac{P(X \leq x, y_1 < Y < y_2)}{P(y_1 < Y < y_2)}$$

$$= \int_{y_1}^{y_2} \int_{-\infty}^x f_{xy}(x, y) dx dy$$

$$\frac{\int_{y_1}^{y_2} \int_{-\infty}^x f_{xy}(x, y) dx dy}{\int_{y_1}^{y_2} f_y(y) dy}$$

1) If $f_{x,y}(x,y) = \{c(2x+y); 0 < x < 2; 0 < y < 3\}$.

(i) Find 'c' value

(ii) Find $P(x > 1, y \leq 2)$

Sol: (i) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{x,y}(x,y) dx dy = 1$

$$= \int_0^3 \int_0^2 c(2x+y) dx dy = 1$$

$$c \int_0^3 \int_0^2 (2x+y) dx dy = 1$$

$$c \int_0^3 \left[\frac{2x^2}{2} + xy \right]_0^2 dy = 1$$

$$c \int_0^3 [4 + 2y] dy = 1$$

$$c \left[4y + \frac{2y^2}{2} \right]_0^3 = 1$$

$$c [12 + 9] = 1$$

$$c [21] = 1$$

$$\boxed{c = \frac{1}{21}}$$

(ii) $P(x > 1, y \leq 2)$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{x,y}(x,y) dx dy$$

$$= \frac{1}{21} \int_0^2 \int_1^2 (2x+y) dx dy$$

$$= \frac{1}{21} \int_0^2 \left[\frac{2x^2}{2} + xy \right]_1^2 dy$$

$$= \frac{1}{21} \int_0^2 (4 + 2y) dy - [1 - y] dy$$

$$= \frac{1}{21} \left[4y + \frac{2y^2}{2} \right]_0^2 - \left[1 - y \right]_0^2 = \frac{12}{21}$$

$$= \frac{1}{21} \int_0^2 (y+3) dy$$

$$= \frac{1}{21} \left[\frac{y^2}{2} + 3y \right]_0^2$$

$$= \frac{1}{21} \left[\frac{4}{2} + 6 \right]$$

$$= \frac{8}{21}$$

2) Find the Marginal densities of x and y given below the joint density functions

$$f_{x,y}(x,y) = \frac{1}{12} u(x) u(y)$$

$$f_{x,y}(x,y) = \frac{1}{12} u(x) u(y) e^{-x/3} e^{-y/4}$$

Sol: $f_x(x) = \int_{-\infty}^{\infty} \frac{1}{12} u(x) u(y) e^{-x/3} e^{-y/4} dy$

$$f_y(y) = \int_{-\infty}^{\infty} \frac{1}{12} u(x) u(y) e^{-x/3} e^{-y/4} dx$$

$$f_{x,y}(x,y) = \frac{1}{12} u(x) u(y) e^{-x/3} e^{-y/4}$$

Now,

$$f_x(x) = \int_{-\infty}^{\infty} \frac{1}{12} u(x) u(y) e^{-x/3} e^{-y/4} dy$$

$$= \frac{1}{12} u(x) e^{-x/3} \int_{-\infty}^{\infty} u(y) e^{-y/4} dy$$

$$= \frac{1}{12} u(x) e^{-x/3} \int_0^{\infty} e^{-y/4} dy$$

$$= \frac{1}{12} u(x) e^{-x/3} \left[\frac{e^{-y/4}}{-1/4} \right]_0^{\infty}$$

$$= \frac{4}{12} u(x) e^{-x/3} [e^{-\infty} - e^0]$$

$$= \frac{1}{3} u(x) e^{-x/3} [-1]$$

$$f_x(x) = \frac{1}{3} u(x) e^{-x/3}$$

$$f_y(y) = \int_{-\infty}^{\infty} \frac{1}{12} u(x) u(y) e^{-x/3} e^{-y/4} dx$$

$$= \frac{1}{12} u(y) e^{-y/4} \int_{-\infty}^{\infty} u(x) e^{-x/3} dx$$

$$= \frac{1}{12} u(y) e^{-y/4} \int_0^{\infty} e^{-x/3} dx$$

$$= \frac{1}{12} u(y) e^{-y/4} \left[\frac{e^{-x/3}}{-1/3} \right]_0^{\infty}$$

$$= -\frac{3}{12} u(y) e^{-y/4} [e^{-\infty} - e^0]$$

$$= -\frac{1}{4} u(y) e^{-y/4} [-1]$$

$$\boxed{f_y(y) = \frac{1}{4} u(y) e^{-y/4}}$$

3) Find the marginal density function of x and y using joint density function $f_{x,y}(x,y) = 2u(x)u(y)e^{-(4y+x/2)}$

Sol:

$$f_{x,y}(x,y) = 2u(x)u(y)e^{-4y} \cdot e^{-x/2}$$

$$f_x(x) = \int_{-\infty}^{\infty} 2u(x)u(y)e^{-4y} e^{-x/2} dy$$

$$= 2u(x)e^{-x/2} \int_{-\infty}^{\infty} u(y)e^{-4y} dy$$

$$= 2u(x)e^{-x/2} \int_0^{\infty} e^{-4y} dy$$

$$= 2u(x)e^{-x/2} \left[\frac{e^{-4y}}{-4} \right]_0^{\infty}$$

$$= -\frac{1}{2} u(x) e^{-x/2} [e^{-\infty} - e^0]$$

$$= -\frac{1}{2} u(x) e^{-x/2} [-1]$$

$$\boxed{f_x(x) = \frac{1}{2} u(x) e^{-x/2}}$$

$$\begin{aligned}
 f_y(y) &= \int_{-\infty}^{\infty} f_{x,y}(x,y) dx \\
 &= \int_{-\infty}^{\infty} 2u(x)u(y) e^{-4y} \cdot e^{-x/2} dx \\
 &= 2u(y) e^{-4y} \int_{-\infty}^{\infty} u(x) e^{-x/2} dx \\
 &= 2u(y) e^{-4y} \int_0^{\infty} e^{-x/2} dx \\
 &= 2u(y) e^{-4y} \left[\frac{e^{-x/2}}{-1/2} \right]_0^{\infty} \\
 &= -4u(y) e^{-4y} (e^{-\infty} - e^0) \\
 &= -4u(y) e^{-4y} (-1)
 \end{aligned}$$

$$f_y(y) = 4u(y) e^{-4y}$$

4) the joint density function is $f_{x,y}(x,y) = a(2x+y^2)$

$0 < x < 2$, $2 < y < 4$. Find the 'a' value.

(ii) $P(X \leq 1, Y > 3)$

Sol: (i) Given $f_{x,y}(x,y) = a(2x+y^2)$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{x,y}(x,y) dx dy = 1$$

$$\int_0^2 \int_2^4 a(2x+y^2) dx dy = 1$$

$$a \int_0^2 \left[\frac{2x^2}{2} + \frac{y^3}{3} \right]_2^4 dy = 1$$

$$a \int_0^2 \left[16 + \frac{64}{3} - 4 - \frac{8}{3} \right] dy = 1$$

$$a \int_0^2 \left(12 + \frac{56}{3} \right) dy = 1$$

$$a \int_0^2 \left(2xy + \frac{y^3}{3} \right) dx = 1$$

$$a \int_0^2 \left(8x + \frac{64}{3} - 4x - \frac{8}{3} \right) dx = 1$$

$$a \int_0^2 \left(4x + \frac{56}{3} \right) dx = 1$$

$$a \left[\frac{4x^2}{2} + \frac{56}{3}x \right]_0^2 = 1$$

$$a \left(2(4) + \frac{56}{3}(2) \right) = 1$$

$$a \left(8 + \frac{112}{3} \right) = 1$$

$$a \left[\frac{24 + 112}{3} \right] = 1$$

$$a \left[\frac{136}{3} \right] = 1$$

$$a = \frac{3}{136} \quad \boxed{a = \frac{3}{136}}$$

(ii) $P(X \leq 1, Y > 3)$

$$= \int_{-\infty}^1 \int_3^{\infty} f_{xy}(x, y) dy dx$$

$$= \int_0^1 \int_3^4 a(2x + y^2) dy dx$$

$$= \frac{3}{136} \int_0^1 \left(2xy + \frac{y^3}{3} \right) dx$$

$$= \frac{3}{136} \int_0^1 \left(8x + \frac{64}{3} - 4x - \frac{27}{3} \right) dx$$

$$= \frac{3}{136} \int_0^1 \left(4x + \frac{37}{3} \right) dx$$

$$= \frac{3}{136} \left[\frac{4x^2}{2} + \frac{37}{3}x \right]_0^1$$

$$\begin{aligned}
 &= -\frac{3}{88} \left(1 - \frac{37}{3} \right) \\
 &= -\frac{3}{88} \left(\frac{3-37}{3} \right) \\
 &= -\frac{3}{88} \left(-\frac{34}{3} \right) \\
 &= \frac{17}{44} //
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{3}{136} \left(\frac{46}{3} \right) \\
 &= \frac{5}{17}
 \end{aligned}$$

5) The joint density function for x and y is

$$f_{x,y}(x,y) = \begin{cases} xy/9, & \text{for } 0 < x < 2, 0 < y < 3 \\ 0 & \text{Otherwise} \end{cases}$$

Find the conditional density functions.

Sol: Given that,

$$f_{x,y}(x,y) = \begin{cases} xy/9, & 0 < x < 2, 0 < y < 3 \\ 0, & \text{Otherwise.} \end{cases}$$

The condition density function of (x,y)

$$f_{x/y}(x/y) = \frac{f_{x,y}(x,y)}{f_y(y)}$$

$$f_{y/x}(y/x) = \frac{f_{y,x}(x,y)}{f_x(x)}$$

$$f_{xy}(x,y) = \frac{\partial^2 f(x,y)}{\partial x \partial y} = (xy)^{-1/2}$$

$$f_y(y) = \int_{-\infty}^{\infty} f_{x,y}(x,y) dx, \quad f_x(x) = \int_{-\infty}^{\infty} f_{x,y}(x,y) dy$$

$$= \int_{-\infty}^{\infty} \frac{xy}{9} dy$$

$$f_x(x) = \int_{-\infty}^{\infty} \frac{xy}{9} dy$$

$$= \frac{2}{9} \int_0^3 \frac{xy}{2} dy$$

$$= \frac{x}{9} \left[\frac{y^2}{2} \right]_0^3$$

$$= \frac{x}{9} \left[\frac{9}{2} \right]$$

$$\boxed{f_x(x) = \frac{x}{2}}$$

$$f_y(y) = \int_{-\infty}^{\infty} \frac{xy}{9} dx$$

$$= \frac{y}{9} \int_0^2 x dx$$

$$= \frac{y}{9} \left[\frac{x^2}{2} \right]_0^2$$

$$= \frac{y}{9} \left[\frac{4}{2} \right]$$

$$= \frac{2y}{9}$$

$$\boxed{f_y(y) = \frac{2y}{9}}$$

$$F_{x/y}(x/y) = \frac{F_{x,y}(x,y)}{f_y(y)} = \frac{xy}{2} \text{ over the region}$$

$$\boxed{F_{xy}(x,y) = \frac{x}{2}}$$

Differentiating on both sides.

$$\frac{d}{dw} [F_w(w)] = \frac{d}{dw} \left[\int_{-\infty}^w f_y(y) dy \int_{-\infty}^{w-y} f_x(x) dx \right]$$

$$= \int_{-\infty}^w f_y(y) dy \cdot \frac{d}{dw} \left[\int_{-\infty}^{w-y} f_x(x) dx \right]$$

$$= \int_{-\infty}^w f_y(y) dy [f_x(x)]_{-\infty}^{w-y}$$

$$= \int_{-\infty}^w f_y(y) f_x(w-y) dy$$

Central Limit Theorem:

→ Central limit theorem says that the probability distribution function of sum of large no. of random variables approaches a gaussian distribution.

→ The distribution of sum of random variables is generally considered to be their individual distribution. However if a random variable is the sum of no. of random variables, then as per central limit theorem. This theorem is assumed to approach gaussian distribution random variables.

Condition-1:

Unequal Distribution:

Let $\bar{x}_i$ and $\sigma_{x_i}^2$ be the mean and variance respectively n random variables, where ' x_i '; $i = 1, 2, 3, \dots, N$ which may have arbitrary probability densities.

→ The central limit theorem states that

$$Y_N = x_1 + x_2 + \dots + x_N \text{ which has mean}$$

$$\bar{Y}_N = \bar{x}_1 + \bar{x}_2 + \dots + \bar{x}_N \text{ and variance}$$

$\sigma_{Y_N}^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 + \dots + \sigma_{x_N}^2$ and probability distribution asymptotically gaussian has n tends to ∞ .

Condition-2:

Equal distribution:

Let 'n' random variables have some distribution and density function and let $Y = x_1 + x_2 + x_3 + \dots + x_n$ also w be the normalized random variable.

$$Y = x_1 + x_2 + x_3 + \dots + x_n$$

$$Y_N = \sum_{n=1}^N x_n$$

$$\bar{Y} = \bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \dots + \bar{x}_n$$

$$\bar{Y}_N = \sum_{n=1}^N \bar{x}_n$$

$$\sigma_Y^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 + \sigma_{x_3}^2 + \dots + \sigma_{x_n}^2$$

$$\sigma_{Y_N}^2 = \sum_{n=1}^N \sigma_{x_n}^2$$

Let w be the normalised gaussian

$$w = \frac{Y - \bar{Y}}{\sigma_Y^2}$$

$$w = \frac{\sum_{n=1}^N x_n - \sum_{n=1}^N \bar{x}_n}{\sum_{n=1}^N (\sigma_{x_n}^2)^{1/2}}$$

$$= \frac{\sum_{n=1}^N (x_n - \bar{x}_n)}{(N \sigma_x^2)^{1/2}}$$

$$= \frac{\sum_{n=1}^N (x_n - \bar{x}_n)}{(N \sigma_x^2)^{1/2}}$$

$$(N \sigma_x^2)^{1/2}$$

$$= \frac{\sum_{n=1}^N (x_n - \bar{x}_n)}{\sqrt{N} \sigma_n}$$

$$W = \frac{1}{\sqrt{N} \sigma_n} \sum_{n=1}^N (x_n - \bar{x}_n)$$

8) Marginal Density function

1. Marginal of 'x' = $F_X(x) = \frac{d}{dx} [F_{XY}(x, \infty)]$

$$= \frac{d}{dx} [F_{XY}(x, \infty)] = \int_{-\infty}^{\infty} f_{XY}(x, y) dy$$

Proof:- The Joint CDF of 'X' ~~$F_X(x) = \frac{d}{dx} [F_X(x)]$~~

X & Y = $F_{XY}(x, y)$

$$= \int_{-\infty}^x \int_{-\infty}^y f_{XY}(x, y) dx dy$$

$$F_{XY}(x, y) = \int_{-\infty}^x \int_{-\infty}^y f_{XY}(x, y) dx dy$$

marginal CDF

~~$$F_{XY}(x, y) = \int_{-\infty}^x \int_{-\infty}^y f_{XY}(x, y) dx dy$$~~

$$\frac{d}{dx} [F_{XY}(x, \infty)] = \frac{d}{dx} \int_{-\infty}^x \int_{-\infty}^{\infty} f_{XY}(x, y) dy dx$$

$$= \int_{-\infty}^{\infty} f_{XY}(x, y) dy - \int_{-\infty}^{\infty} f_{XY}(x, \infty) dy$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy$$

UNIT - 3
* Define the expected value of the function of joint random variables.

Let $g(x, y)$ be the function of two random variables x and y with joint density function $f_{xy}(x, y)$ then the expected value of function is given by,

$$\bar{g} = E[g(x, y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{xy}(x, y) dx dy$$

Joint Moments:

1) Joint Moment about Origin:

consider two random variables x and y then joint moment about origin is defined as the expected value of $x^n y^k$, mathematically it can be expressed as.

$$M_{nk} = E[x^n y^k] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^n y^k f_{xy}(x, y) dx dy$$

Zeroth Order Moment:

If $n=0, k=0$

$$\begin{aligned} M_{00} &= E[x^0 y^0] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^0 y^0 f_{xy}(x, y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{xy}(x, y) dx dy \\ &= \int_{-\infty}^{\infty} f_y(y) dy \quad [\because \int_{-\infty}^{\infty} f_{xy}(x, y) dx = f_y(y)] \\ &= 1 \end{aligned}$$

First Order moment:

Case-1: If $n=0, k=1$

$$M_{01} = E[x^0 y^1] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^0 y f_{xy}(x, y) dx dy$$

$$\begin{aligned}
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_{xy}(x, y) dx dy \\
 &= \int_{-\infty}^{\infty} y f_y(y) dy \quad \left[\because \int_{-\infty}^{\infty} f_{xy}(x, y) dx = f_y(y) \right] \\
 &= E[Y] \\
 &= \bar{y}
 \end{aligned}$$

case-2: If $n=1, k=0$

$$M_{nk} = E[x^1 y^0] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x y^0 f_{xy}(x, y) dx dy$$

$$= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f_{xy}(x, y) dy dx$$

$$= \int_{-\infty}^{\infty} x f_x(x) dx \quad \left[\because \int_{-\infty}^{\infty} f_{xy}(x, y) dy = f_x(x) \right]$$

$$= E[x]$$

$$= \bar{x}$$

second order Moment: (M_{02}, M_{20}, M_{11})

case-1: If $n=0, k=2$

$$M_{nk} = E[x^0 y^2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^0 y^2 f_{xy}(x, y) dx dy$$

$$= \int_{-\infty}^{\infty} y^2 \int_{-\infty}^{\infty} f_{xy}(x, y) dx dy$$

$$= \int_{-\infty}^{\infty} y^2 f_y(y) dy$$

$$= E[y^2] \text{ --- Mean square Error.}$$

case-2: If $n=2, k=0$

$$M_{nk} = E[x^2 y^0] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 y^0 f_{xy}(x, y) dx dy$$

$$= \int_{-\infty}^{\infty} x^2 \int_{-\infty}^{\infty} f_{xy}(x, y) dy dx$$

$$= \int_{-\infty}^{\infty} x f_x(x) dx = E[x]$$

$$= \int_{-\infty}^{\infty} x^2 f_x(x) dx = E[x^2]$$

$$= E[x^2]$$

Case-3: If $n=1, k=1$

$$M_{nk} = E[x'y'] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x'y' f_{xy}(x,y) dx dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{xy}(x,y) dx dy$$

$$= E[xy]$$

2) central joint moment about mean (a) central moment:

let us consider two random variables x and y then the joint central moment is defined as expected value of $(x-\bar{x})^n (y-\bar{y})^k$, mathematically it can be represented as

$$M_{nk} = E[(x-\bar{x})^n (y-\bar{y})^k] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-\bar{x})^n (y-\bar{y})^k f_{xy}(x,y) dx dy$$

case-1:

Zeroth order central moment:

case-1: If $n=0, k=0$

$$\mu_{00} = E[(x-\bar{x})^0 (y-\bar{y})^0] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-\bar{x})^0 (y-\bar{y})^0 f_{xy}(x,y) dx dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{xy}(x,y) dx dy$$

$$= 1$$

First order central moment:

case-1: If $n=0, k=1$

$$\mu_{01} = E[(x-\bar{x})^0 (y-\bar{y})^1] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-\bar{x})^0 (y-\bar{y})^1 f_{xy}(x,y) dx dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (y-\bar{y}) f_{xy}(x,y) dx dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (y - \bar{y}) f_{xy}(x, y) dx dy$$

$$= \int_{-\infty}^{\infty} (y - \bar{y}) E[f_{xy}(x, y)] dy$$

$$= \bar{y} - \bar{y}$$

$$= 0$$

case-2: If $n=1, k=0$

$$\mu_{10} = E[(x - \bar{x})(y - \bar{y})^0] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \bar{x})(y - \bar{y})^0 f_{xy}(x, y) dx dy$$

$$= \int_{-\infty}^{\infty} (x - \bar{x}) f_x(x) dx$$

$$= E[x] - \bar{x}$$

$$= \bar{x} - \bar{x}$$

$$= 0$$

case-3: If $n=1, k=1$

$$\mu_{11} = E[(x - \bar{x})(y - \bar{y})'] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \bar{x})(y - \bar{y})' f_{xy}(x, y) dx dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \bar{x})(y - \bar{y}) f_{xy}(x, y) dx dy$$

→ Define covariance.

let x and y be the two random variables

the second order joint central moment is called co-variance.

$$\boxed{\mu_{11} = E[(x - \bar{x})(y - \bar{y})] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \bar{x})(y - \bar{y}) f_{xy}(x, y) dx dy}$$

properties:

1) If x and y are two random variables then the co-variance is -

$$C_{xy} = R_{xy} - \bar{x}\bar{y}$$

$$= \text{correlation} - E[x]E[y]$$

$$C_{xy} = E[(x - \bar{x})(y - \bar{y})]$$

$$= E[xy - x\bar{y} - \bar{x}y + \bar{x}\bar{y}]$$

$$= E[xy] - E[x\bar{y}] - E[\bar{x}y] + E[\bar{x}\bar{y}]$$

$$= R_{xy} - \bar{y}E[x] - \bar{x}E[y] + \bar{x}\bar{y}$$

$$= R_{xy} - \bar{y}\bar{x} - \bar{x}\bar{y} + \bar{x}\bar{y}$$

$$\boxed{C_{xy} = R_{xy} - \bar{y}\bar{x}}$$

2) If two random variable x and y are independent the the co-variance always becomes zero.

WKT,

$$C_{xy} = R_{xy} - \bar{x}\bar{y}$$

$$= E[xy] - \bar{x}\bar{y}$$

$$= E[x]E[y] - \bar{x}\bar{y}$$

$$= \bar{x}\bar{y} - \bar{x}\bar{y}$$

3) For the two random Variables x and y then

a) $\text{Var}(x+y) = \text{Var}(x) + \text{Var}(y) + 2C_{xy}$

b) $\text{Var}(x-y) = \text{Var}(x) + \text{Var}(y) - 2C_{xy}$

WKT,

$$\text{a) } \text{Var}(x+y) = \sigma_{(x+y)}^2 = E[(x+y)^2] - E[(x+y)]^2$$

$$= E[x^2 + y^2 + 2xy] - [E[x]^2 + E[y]^2 + 2E[x]E[y]]$$

$$= E[x^2] + E[y^2] + 2E[xy] - E[x]^2 - E[y]^2 - 2E[x]E[y]$$

$$= E[x^2] - E[x]^2 + E[y^2] - E[y]^2 + 2[E[xy] - E[x]E[y]]$$

$$= \sigma_x^2 + \sigma_y^2 + 2(R_{xy} - \bar{x}\bar{y})$$

$$= \sigma_x^2 + \sigma_y^2 + 2C_{xy}$$

$$\boxed{\text{var}(x+y) = \text{Var}(x) + \text{Var}(y) + 2C_{xy}}$$

$$\begin{aligned} \text{b) } \text{var}(x-y) &= \sigma_{(x-y)}^2 = E[(x-y)^2] - E[(x-y)]^2 \\ &= E[x^2 + y^2 - 2xy] - [E[x]^2 + E[y]^2 - 2E[x]E[y]] \\ &= E[x^2] + E[y^2] - 2E[xy] - E[x]^2 - E[y]^2 + 2E[x]E[y] \\ &= E[x^2] - E[x]^2 + E[y^2] - E[y]^2 - 2[E[xy] - E[x]E[y]] \\ &= \sigma_x^2 + \sigma_y^2 - 2[R_{xy} - \bar{x}\bar{y}] \\ &= \sigma_x^2 + \sigma_y^2 - 2C_{xy} \end{aligned}$$

$$\boxed{\text{var}(x-y) = \text{Var}(x) + \text{Var}(y) - 2C_{xy}}$$

4) If x and y be any two random variables then the covariance of $(x+a)$, $(y+b)$ where a, b are any real constants then covariance of

$$\text{cov}(x+a, y+b) = \text{cov}(x, y) = C_{xy}$$

$$\text{WKT, } C_{xy} = R_{xy} - \bar{x}\bar{y}$$

$$\text{covariance} = E[(x-\bar{x})(y-\bar{y})]$$

$$= E[(x+a-\bar{x}+a)(y+b-\bar{y}+b)]$$

$$= E[(x-\bar{x})(y-\bar{y})]$$

$$= \text{Cov}(x, y)$$

$$= C_{xy}$$

5) If x and y are any two random variables, then the covariance of ax, by is

$$\text{cov}(ax, by) = abc_{xy}$$

$$\text{Covariance}_{(x,y)} = E[(x-\bar{x})(y-\bar{y})]$$

$$\text{cov}(ax, by) = E[(a(x-\bar{x}))(b(y-\bar{y}))]$$

$$= E[a(x-\bar{x})b(y-\bar{y})]$$

$$= abE[(x-\bar{x})(y-\bar{y})]$$

$$\boxed{\text{Cov}(ax, by) = abc_{xy}}$$

6) If x and y, z are any three random variables then the covariance of $(x+y, z)$ is

$$\boxed{\text{Cov}(x+y, z) = \text{Cov}(x, z) + \text{Cov}(y, z)}$$

$$\overline{x+y} = \bar{x} + \bar{y}$$

$$\text{Cov}(x+y, z) = E[(x+y - \overline{x+y})(z - \bar{z})]$$

$$= E[xz - x\bar{z} + yz - y\bar{z} - \bar{x}z + \bar{x}\bar{z} - \bar{y}z + \bar{y}\bar{z}]$$

$$= E[xz] - \bar{z}E[x] + E[yz] - \bar{z}E[y] - \bar{x}E[z] + E[\bar{x}\bar{z}] - \bar{y}E[z] + E[\bar{y}\bar{z}]$$

$$= E[xz] - \bar{x}\bar{z} + E[yz] - \bar{y}\bar{z} - \bar{x}\bar{z} + \bar{x}\bar{z} - \bar{y}\bar{z} + \bar{y}\bar{z}$$

$$= E[xz] + E[yz] - \bar{x}\bar{z} - \bar{y}\bar{z}$$

$$= R_{xz} - \bar{x}\bar{z} + R_{yz} - \bar{y}\bar{z}$$

$$= C_{xz} + C_{yz}$$

$$= \text{Cov}(x, z) + \text{Cov}(y, z)$$

Joint characteristic function: $\phi_{xy}(\omega_1, \omega_2)$ (properties)

→ If x, y are two random variables, the expected value of $E[e^{j\omega_1 x + j\omega_2 y}]$ is called joint characteristic function. $f_{xy}(x, y)$ is the joint density function, then.

$$\phi_{xy}(\omega_1, \omega_2) = E[e^{j\omega_1 x + j\omega_2 y}]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{j\omega_1 x + j\omega_2 y}) f_{xy}(x, y) dx dy$$

$$\text{If } \omega_2 = 0$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j\omega_1 x + 0} f_{xy}(x, y) dx dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j\omega_1 x} f_{xy}(x, y) dx dy$$

$$= \phi_x(\omega_1)$$

$$E[e^{j\omega_1 x + j\omega_2 y}] = \phi_{xy}(\omega_1, \omega_2)$$

$$E[e^{j\omega_1 x + 0}] = \phi_{xy}(\omega_1, 0)$$

$$E[e^{j\omega_1 x}] = \phi_{xy}(\omega_1)$$

$$\boxed{\phi_x(\omega_1) = \phi_{xy}(\omega_1)}$$

$$\text{If } \omega_1 = 0$$

$$\phi_{xy}(0, \omega_2) = E[e^{0 + j\omega_2 y}]$$

$$\phi_{xy}(\omega_2) = E[e^{j\omega_2 y}]$$

$$\boxed{\phi_{xy}(\omega_2) = \phi_y(\omega_2)}$$

Here, $\phi_x(\omega_1)$, $\phi_y(\omega_2)$ generally called Marginal joint characteristic function.

→ If x and y are independent, $f_{xy}(x,y)$

$$\phi_{xy}(\omega_1, \omega_2) = E[e^{j\omega_1 x + j\omega_2 y}]$$

$$= E[e^{j\omega_1 x} \cdot e^{j\omega_2 y}]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j\omega_1 x} \cdot e^{j\omega_2 y} f_{xy}(x,y) dx dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j\omega_1 x} e^{j\omega_2 y} f_x(x) \cdot f_y(y) dx dy$$

$$= \int_{-\infty}^{\infty} e^{j\omega_1 x} f_x(x) dx \int_{-\infty}^{\infty} e^{j\omega_2 y} f_y(y) dy$$

$$\boxed{\phi_{xy}(\omega_1, \omega_2) = \phi_x(\omega_1) \cdot \phi_y(\omega_2)}$$

1) A Random process is describe by $x(t) = a$, where 'a' is continuous Random variable and uniformly distributed in (0,1) then show that the random process $x(t)$ is stationary.

$$f_A(A) = 1 \quad [\text{uniform distribution}]$$

$$x(t) = A$$

$$E[x(t)] = \int_{-\infty}^{\infty} x(t) f_x(x) dx$$

$$= \int_0^1 A f_A(A) dA$$

$$= \int_0^1 A (1) dA = \frac{A^2}{2} \Big|_0^1 = \frac{1}{2}$$

∴ first condition is satisfied.

$R_{xx}(\tau)$ is independent of time.

$$R_{xx}(\tau) = E[x(t)x(t+\tau)] \quad \text{--- ①}$$

Given random process is stationary then

$$x(t) = x(t+\tau) \quad \text{--- ②}$$

Substitute ② in ①

$$= E[x(t)x(t)]$$

$$\begin{aligned}
 &= E[x^2(t)] \\
 &= \int_{-\infty}^{\infty} x^2(t) f_x(x) dx \\
 &= \int_0^1 x^2 + A(A) dx \\
 &= \left[\frac{x^3}{3} \right]_0^1 = 1/3
 \end{aligned}$$

It is independent of time

$\therefore$ It is a stationary process.

2) A wide sense stationary random process has an autocorrelation function of form $R_{xx}(\tau) = 2e^{-2|\tau|} + 4e^{-4|\tau|}$. Find mean value and mean square value?

$$R_{xx}(\tau) = 2e^{-2|\tau|} + 4e^{-4|\tau|}$$

From the properties of Autocorrelation

$$\text{At } R_{xx}(\tau) = \bar{x}^2$$

$$\text{At } 2e^{-2|\tau|} + 4e^{-4|\tau|}$$

$$\text{At } 2e^{-2|\infty|} + 4e^{-4|\infty|}$$

$$0 + 0 = 0$$

$$\bar{x}^2 = 0$$

$$\boxed{\bar{x} = 0}$$

$$R_{xx}(0) = E[x^2(t)]$$

$$\begin{aligned}
 R_{xx}(0) &= 2e^{-2|0|} + 4e^{-4|0|} \\
 &= 2e^{-2 \cdot 0} + 4e^{-4 \cdot 0} \\
 &= 2 + 4 = 6
 \end{aligned}$$

$$\sigma_x^2 = E[x^2(t)] - E[x(t)]^2$$

$$= 6 - 0 = 6$$

$$\boxed{\sigma_x^2 = 6}$$

3) The random process $z(t) = A \cos \omega_0 t + B \sin \omega_0 t$ where A and B are Gaussian Random Variables with zero mean and Variance (σ_A^2) show that $z(t)$ is Wide sense stationary.

Sol: Given that,

$$z(t) = A \cos \omega_0 t + B \sin \omega_0 t$$

$$\text{Mean} = 0$$

A process is said to be wide sense stationary process if it satisfies.

$$\rightarrow E[z(t)] = \bar{x} = \text{constant}$$

$$\rightarrow R_{xx}(\tau) \text{ is independent of time.}$$

Condition -1:

$$E[z(t)] = E[A \cos \omega_0 t + B \sin \omega_0 t]$$

$$= E[A] \cos \omega_0 t + E[B] \sin \omega_0 t$$

$$= 0 (\cos \omega_0 t) + 0 (\sin \omega_0 t)$$

$$= 0$$

$$= \text{constant. (It satisfies condition -1)}$$

Condition -2:

The Autocorrelation function of random process $z(t)$

$$R_{xx}(\tau) = E[z(t)z(t+\tau)]$$

$$= E[(A \cos \omega_0 t + B \sin \omega_0 t)(A \cos \omega_0 (t+\tau) +$$

$$- B \sin \omega_0 (t+\tau))]$$

$$= E[A^2 \cos \omega_0 t \cos \omega_0 (t+\tau) + AB \sin \omega_0 t \cos \omega_0 (t+\tau) +$$

$$- AB \cos \omega_0 t \sin \omega_0 (t+\tau) + B^2 \sin \omega_0 t \sin \omega_0 (t+\tau)]$$

$$= E[A^2] \cos \omega_0 t \cos \omega_0 (t+\tau) + E[AB] \sin (2\omega_0 t + \omega_0 \tau) +$$

$$E[B^2] \sin \omega_0 t \sin \omega_0 (t+\tau)$$

A and B are uncorrelated zero mean value random variable.

$$\therefore E[AB] = 0$$

A and B have same Variance.

$$E[A^2] = E[B^2] = \sigma^2$$

$$\begin{aligned} \therefore R_{xx}(\tau) &= \sigma^2 \cos \omega_0 t \cos(t + \tau) + 0 + \sigma^2 \sin \omega_0 t \sin(t + \tau) \\ &= \sigma^2 \cos(\omega_0 t - \omega_0 t - \omega_0 \tau) \\ &= \sigma^2 \cos(-\omega_0 \tau) \\ &= \sigma^2 \cos \omega_0 \tau \rightarrow \text{Independent of time} \end{aligned}$$

→ It satisfies the condition - 2

∴ Given random process $x(t)$ satisfies the two conditions.

→ It is wide sense stationary process.

4) consider a random process $x(t) = A \cos \omega t$, A is uniform distribution function, where 'w' is constant and 'A' is a random variable, uniformly distributed over (0,1). Find the autocorrelation and Auto-covariance of $x(t)$.

Sol: Given Random process is

$$x(t) = A \cos \omega t$$

∴ A is uniform distribution function (0,1)

$$f_A(A) = \begin{cases} 1 & 0 < A < 1 \\ 0 & \text{otherwise} \end{cases}$$

Autocorrelation function of $x(t)$

$$R_{xx}(t, t + \tau) = E[x(t)x(t + \tau)]$$

$$= \int_0^1 A \cos \omega t \cdot A \cos \omega(t+\tau) \cdot 1 \cdot dA$$

$$= \int_0^1 A^2 \cos \omega t \cos \omega(t+\tau) dA$$

$$= \cos \omega t \cos \omega(t+\tau) \int_0^1 A^2 dA$$

$$= \cos \omega t \cos \omega(t+\tau) \left[\frac{A^3}{3} \right]_0^1$$

$$= \frac{1}{3} \cos \omega t \cos \omega(t+\tau)$$

$$\therefore R_{xx}(t, t+\tau) = \frac{1}{3} \cos \omega t \cdot \cos \omega(t+\tau)$$

Autocovariance function of $x(t)$:

$$C_{xx}(t, t+\tau) = R_{xx}(t, t+\tau) - E[x(t)]E[x(t+\tau)]$$

$$E[x(t)] = \int_0^1 x(t) f_A(A) dA$$

$$= \int_0^1 A \cos \omega t \cdot 1 \cdot dA$$

$$= \cos \omega t \left[\frac{A^2}{2} \right]_0^1$$

$$E[x(t)] = \frac{\cos \omega t}{2}$$

$$E[x(t+\tau)] = \cos \omega t \cos \omega(t+\tau)$$

$$= \frac{\cos \omega t}{2} \cdot \frac{\cos \omega(t+\tau)}{2}$$

$$= \frac{\cos \omega t \cdot \cos \omega(t+\tau)}{4}$$

$$= \frac{\cos \omega t \cdot \cos \omega(t+\tau)}{12}$$

$$\therefore C_{xx}(t, t+\tau) = \cos \omega t \cdot \cos \omega(t+\tau)$$

$$12$$

3. Statistically independent, 0 mean Random process of $x(t)$, $y(t)$ have Auto correlation.

$$R_{xx}(\tau) = e^{-|\tau|}, \quad R_{yy}(\tau) = \cos(2\pi\tau)$$

1) Find the autocorrelation of sum $w_1(t) = x(t) + y(t)$

2) Find the autocorrelation function of difference $w_2(t) = x(t) - y(t)$.

3) Find the correlation function of $w_1(t)$ and $w_2(t)$.

Sol: 1) $R_{xx}(\tau) = E[x(t)x(t+\tau)]$

$$R_{w_1}(\tau) = E[w_1(t)w_1(t+\tau)]$$

$$= E[(x(t) + y(t))(x(t+\tau) + y(t+\tau))]$$

$$= E[x(t)x(t+\tau) + x(t)y(t+\tau) + y(t)x(t+\tau) + y(t)y(t+\tau)]$$

$$= E[x(t)x(t+\tau)] + E[x(t)y(t+\tau)] + E[y(t)x(t+\tau)] + E[y(t)y(t+\tau)]$$

$$= R_{xx}(\tau) + R_{xy}(\tau) + R_{yx}(\tau) + R_{yy}(\tau)$$

$$= R_{xx}(\tau) + 2R_{xy}(\tau) + R_{yy}(\tau)$$

$$= R_{xx}(\tau) + R_{yy}(\tau)$$

$$R_{w_1}(\tau) = e^{-|\tau|} + \cos(2\pi\tau)$$

$$= e^{-|\tau|} + \cos(2\pi\tau)$$

2) $R_{w_2}(\tau) = E[w_2(t)w_2(t+\tau)]$

$$= E[(x(t) - y(t))(x(t+\tau) - y(t+\tau))]$$

$$= E[x(t)x(t+\tau) - x(t)y(t+\tau) - y(t)x(t+\tau) + y(t)y(t+\tau)]$$

$$= E[x(t)x(t+\tau)] - E[x(t)y(t+\tau)] - E[y(t)x(t+\tau)] + E[y(t)y(t+\tau)]$$

$$= R_{xx}(\tau) - E[x(t)y(t+\tau)] - E[y(t)x(t+\tau)] + R_{yy}(\tau)$$

$$= R_{xx}(\tau) - 0 - 0 + R_{yy}(\tau)$$

value of

$$= R_{xx}(\tau) - R_{xy}(\tau) + R_{yx}(\tau) + R_{yy}(\tau)$$

$$= R_{xx}(\tau) - 2R_{xy}(\tau) + R_{yy}(\tau)$$

$$= R_{xx}(\tau) + R_{yy}(\tau)$$

$$= e^{-|\tau|} + \cos 2\pi\tau$$

UNIT - IV RANDOM PROCESS

(STOCHASTIC PROCESS)

- The Random process are also called as stochastic process which deals with randomly varying time wave forms such as any message signals and noise signals.
- probability distribution and probability density function give the completely statistical characteristics of random signal.
- A Random process is a function of both sample space and time variables. It can be represented as $\{X(s, t)\}$.
- Random variable as a function of time is called Random process.
- Random variable indexed by time is also called Random process.

classification of Random process:

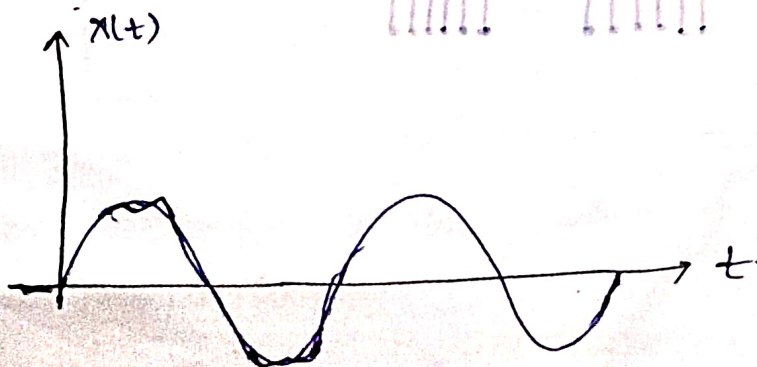
- Random process mainly classified into 4 types.

- (1) continuous Random process
- (2) Discrete Random process
- (3) Continuous Random Sequence
- (4) Discrete Random Sequence.

(1) continuous Random process: (x -continuous, t -continuous)

- A Random process is said to be continuous if random variable 'x' and time 't' are continuous is called continuous Random process.

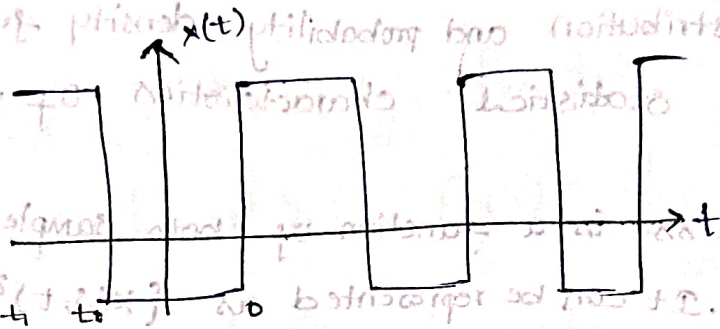
Eg.:



Discrete Random process: (t -continuous, x -discrete)

→ A Random process in discrete Random process, the random variable ' x ' has only discrete values while time ' t ' is continuous. A digital encoded signal has only two discrete values, i.e. positive level and negative level but time ' t ' is continuous.

Ex:

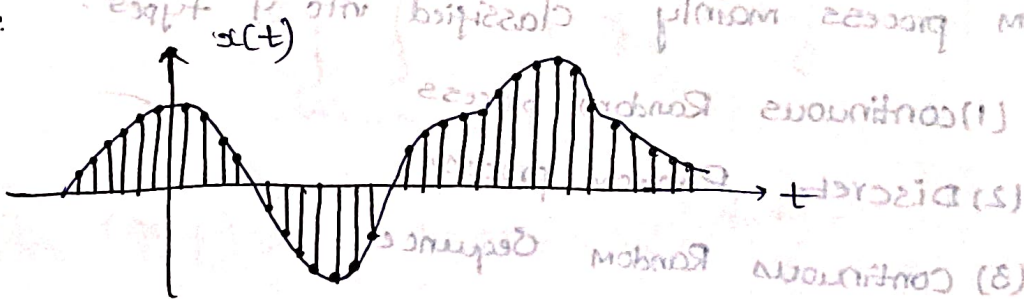


Continuous Random Sequence: (x -continuous, t -discrete)

→ A Random process for which the Random Variable ' x ' is continuous but ' t ' has discrete values is called continuous Random sequence.

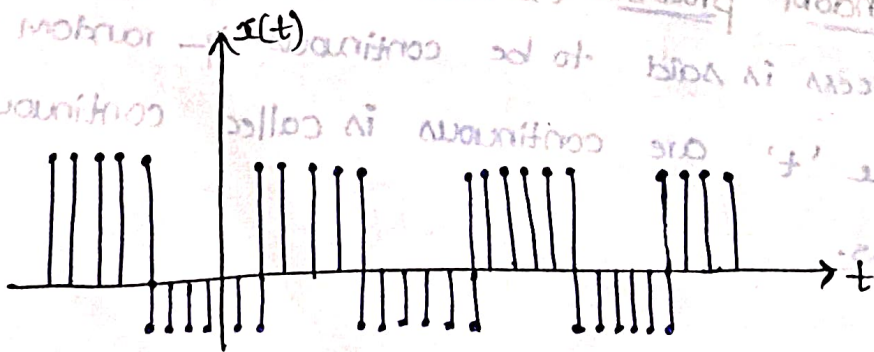
→ A continuous Random signal is defined only at discrete time intervals is called discrete time Random process.

Ex:



Discrete Random Sequence: (x -discrete, t -discrete)

→

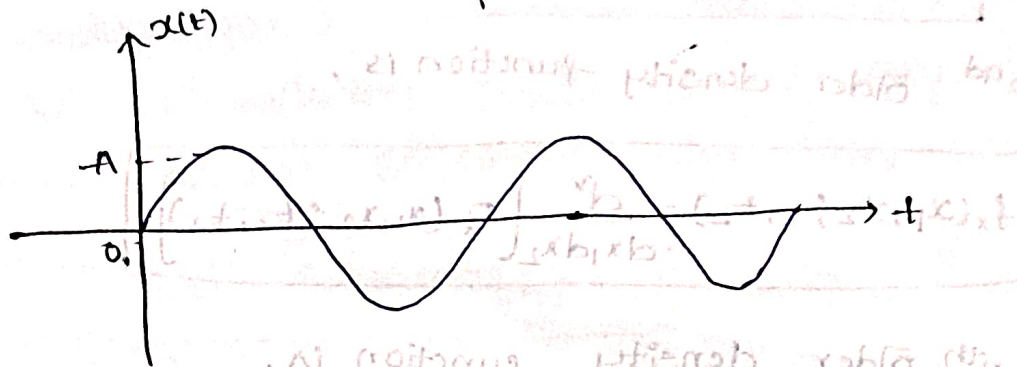


Deterministic (a) Non-Deterministic:

Deterministic Random process:

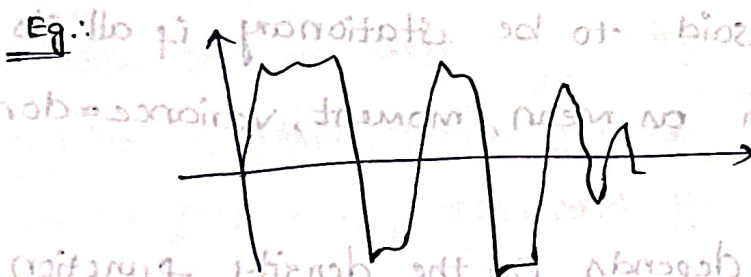
A process in which its future values of any sample function can be predicted from its past values.

Eg.: $x(t) = A \sin(\omega t + \theta)$



Non-Deterministic Random process:

A process in which its future values of any sample function can not be predicted from its past values.



probability distribution and density function of Random

Process:

→ Distributive and density function plays vital role in defining a stationary random process.

→ The distributive function of random variable 'x' at a specific time 't' is defined as,

$$F_x(x; t) = P(X(t) \leq x)$$

→ The second order joint distribution function of two random variables x_1, x_2 with time t_1 and t_2 can be defined as,

$$F_x(x_1, x_2; t_1, t_2) = P(X(t_1) \leq x_1, X(t_2) \leq x_2)$$

→ The n^{th} order joint distribution function is,

$$F_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = P(X(t_1) \leq x_1, X(t_2) \leq x_2, \dots, X(t_n) \leq x_n)$$

→ The 1st order density function is,

$$f_X(x, t) = \frac{d}{dx} [F_X(x, t)]$$

→ The 2nd order density function is,

$$f_X(x_1, x_2; t_1, t_2) = \frac{d^2}{dx_1 dx_2} [F_X(x_1, x_2; t_1, t_2)]$$

→ The n^{th} order density function is,

$$f_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = \frac{d^n}{dx_1 dx_2 \dots dx_n} [F_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n)]$$

Concept of Stationary Random process:-

→ A Random process is said to be stationary if all its statistical properties such as mean, moment, variance etc. do not change with time.

→ The stationary which depends on the density function has different levels (a) orders.

First Order Stationary Random process: ($E[X(t)] \rightarrow \text{constant}$)

→ A Random process is said to be stationary to order 1 (a)

first order stationary,

→ If its first order density function do not change with time (a) shift in time values.

$$E[X(t)] = \bar{X} = \text{Constant}$$

$$f_X(x; t) = f_X(x; t + \Delta t)$$

Second Order Stationary Random process: ($R_{XX}(\tau) \rightarrow \text{Independent of time}$)

→ A Random process is said to be stationary to order 2

(a) Second order stationary if its second order joint

density function do not change with time (3) Shift in the values, i.e.,

$$f_x(x_1, x_2; t_1, t_2) = f_x(x_1, x_2; t_1 + \Delta t_1, t_2 + \Delta t_2) \quad \forall t_1, t_2 \text{ and } \Delta t_1$$

→ It is a function of time difference $t_2 - t_1$ and not absolute time t_1 .

→ The condition for a process to be second order stationary is that its autocorrelation should depend only on time difference and not on absolute time. ($R_{xx}(t_1, t_2)$)

$$R_{xx}(t_1, t_2) = E[x(t_1)x(t_2)]$$

$$R_{xx}(\tau) = E[x(t)x(t+\tau)]$$

→ $R_{xx}(\tau)$ should be independent of time.

Statistical Independence:

→ If Random variables $x(t_1), x(t_2), x(t_3), \dots, x(t_n)$ of

Random process $x(t)$ are independent to the Random variables $y(t_1), y(t_2), y(t_3), \dots, y(t_n)$ of another Random process $y(t)$ then, $x(t)$ and $y(t)$ are said to be statistically Independence.

→ Joint density function of two statistically independent Random variables,

$$f_{xy}(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n; t_1, t_2, \dots, t_n; t_1, t_2, t_3, \dots, t_n)$$

$$= f_x(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) f_y(y_1, y_2, y_3, \dots, y_n; t_1, t_2, \dots, t_n)$$

Wide-sense stationary process:

If a Random process $x(t)$ is a second order stationary process then it is called as wide sense stationary (3) weak-sense stationary process.

Strict sense stationary process [sss]:

A random process $x(t)$ is said to be strict sense stationary if its n th order joint density function do not change with time (or) shift in time.

$$f_x(x_1, x_2, x_3, \dots, x_n; t_1, t_2, t_3, \dots, t_n) =$$

$$f_x(x_1, x_2, x_3, \dots, x_n; t_1 + \Delta t_1, t_2 + \Delta t_2, \dots, t_n + \Delta t_n)$$

Time Average:

consider a Random process $x(t)$. Let $x(t)$ be a sample function which exists for all time at a fixed value in the given sample space 's'. The Average value of $x(t)$ taken over all time is called Time Average of $x(t)$. It is also called mean value of $x(t)$.

$$A[x] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) dt$$

Time Autocorrelation function:

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) x(t+\tau) dt$$

$$\langle R_{xx}(\tau) \rangle = \overline{R_{xx}(\tau)}$$

Time Cross-correlation:

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) y(t+\tau) dt$$

$$\langle R_{xy}(\tau) \rangle = \overline{R_{xy}(\tau)}$$

Ergodic Random process:

Ergodicity is defined as the future or characteristic of an ergodic random process.

A random process is said to be ergodic if it satisfies the two conditions.

→ The time average and statistical average must and should equal.

$$\bar{x} = \bar{\bar{x}}$$

$$E[x(t)] = A[x(t)]$$

→ The time auto correlation function and auto correlation function must and should equal.

$$R_{xx}(\tau) = \overline{R_{xx}(\tau)}$$

$$R_{xx}(\tau) = E[x(t)x(t+\tau)]$$

$$\overline{R_{xx}(\tau)} = A[x(t)x(t+\tau)]$$

$$R_{xx}\tau = A[x(t)x(t+\tau)]$$

properties of Auto correlation:

The auto-correlation function defined as the function that gives the relation between time delayed and undelayed version of same random process.

$$R_{xx}(\tau) = E[x(t)x(t+\tau)]$$

$$R_{xx}(t_1, t_2) = (E[x(t_1)x(t_2)])$$

$$\begin{cases} \tau = t_2 - t_1 \\ t_2 = \tau + t_1 \end{cases}$$

$$\therefore R_{xx}(t_1, t_2) = R_{xx}(\tau)$$

properties:

1) The Mean square value of $x(t)$ is $E[x^2(t)] = R_{xx}(0)$ is the power of the Random process.

Sol. From the autocorrelation definition,

$$R_{xx}(\tau) = E[x(t)x(t+\tau)]$$

$$R_{xx}(0) = E[x(t)x(t+0)]$$

$$= E[x(t)x(t)]$$

$$R_{xx}(0) = E[x^2(t)] //$$

2) The Autocorrelation function maximum $|R_{xx}(\tau)| \leq R_{xx}(0)$.

Sol. consider two random processes, $x(t_1)$ and $x(t_2)$ defined

at time intervals t_1 and t_2 and consider a positive

quantity $[x(t_1) \pm x(t_2)]^2 \geq 0$

Applying expected value on both sides,

$$E[x(t_1) \pm x(t_2)]^2 \geq E[0]$$

$$E[x^2(t_1) + x^2(t_2) \pm 2x(t_1)x(t_2)] \geq 0$$

$$E[x^2(t_1)] + E[x^2(t_2)] \pm E[2x(t_1)x(t_2)] \geq 0$$

$$R_{xx}(0) + R_{xx}(0) \pm 2R_{xx}(\tau) \geq 0$$

$$2R_{xx}(0) \pm 2R_{xx}(\tau) \geq 0$$

$$\pm R_{xx}(\tau) \geq -R_{xx}(0)$$

$$|R_{xx}(\tau)| \leq R_{xx}(0)$$

$$\therefore |R_{xx}(\tau)| \leq R_{xx}(0)$$

3) Autocorrelation function $R_{xx}(\tau)$ always even function of τ ;
 $R_{xx}(\tau) = R_{xx}(-\tau)$

$$R_{xx}(\tau) = E[x(t)x(t+\tau)]$$

Replace τ by $-\tau$

$$R_{xx}(-\tau) = E[x(t)x(t-\tau)]$$

$$\text{let } t - \tau = 0$$

$$t = 0 + \tau$$

$$= E[x(0+\tau)x(0)]$$

$$= E[x(0)x(0+\tau)]$$

$$R_{xx}(-\tau) = R_{xx}(\tau)$$

4) If a random process $x(t)$ has non-zero mean value.

$E[x(t)] \neq 0$ and Ergodic with no periodic components

then $\lim_{T \rightarrow \infty} R_{xx}(\tau) = \bar{x}^2$

$$\lim_{T \rightarrow \infty} R_{xx}(\tau) = R_{xx}(\tau)$$

$$= E[x(t)x(t+\tau)]$$

$$= E[x(t_1)x(t_2)]$$

$$= E[x(t_1)]E[x(t_2)]$$

$$= E[x(t)]^2$$

$$= \bar{x}^2$$

proof: Consider a random variable $x(t)$ with a random variables $x(t_1)$ and $x(t_2)$ given the mean value $E[x(t)] \neq 0$.

$$R_{xx}(\tau) = \bar{x}^2 + R_{xx}(\tau)$$

5) If $x(t)$ is periodic its auto-correlation-function is also periodic.

proof: consider a random variable $x(t)$ which is periodic with period T_0 , then

$$x(t) = x(t + T_0)$$

$$x(t + \tau + T_0) = x(t + \tau)$$

$$R_{xx}(\tau) = E[x(t)x(t+\tau)]$$

$$R_{xx}(\tau + T_0) = E[x(t)x(t+\tau+T_0)]$$

$$R_{xx}(\tau + T_0) = E[x(t)x(t+\tau)]$$

6) The auto-correlation random process $R_{xx}(\tau)$ cannot have any arbitrary state, The auto-correlation of function is random process.

$R_{xx}(\tau)$ is an even function and has maximum value at origin. Hence, autocorrelation function cannot have an arbitrary constant.

If a Random process $x(t)$ with '0' mean has the dc component 'A' has $y(t) = A + x(t)$ then

$$\begin{aligned} R_{yy}(\tau) &= E[y(t)y(t+\tau)] \\ &= E[(A+x(t))(A+x(t+\tau))] \\ &= E[A^2 + A(x(t)+x(t+\tau)) + x(t)x(t+\tau)] \\ &= E[A^2] + AE[x(t)+x(t+\tau)] + E[x(t)x(t+\tau)] \\ &= A^2 + R_{xx}(\tau) + AE[x(t)+x(t+\tau)] \end{aligned}$$

According to the property of Auto correlation,
 $E[x(t)+x(t+\tau)] \geq 0$

$$\boxed{R_{yy}(\tau) = A^2 + R_{xx}(\tau)}$$

* If a Random process $z(t)$ is sum of two random process $x(t)$ and $y(t)$, i.e., $z(t) = x(t) + y(t)$ - then

$$R_{zz}(\tau) = R_{xx}(\tau) + R_{xy}(\tau) + R_{yx}(\tau) + R_{yy}(\tau)$$

$$R_{zz}(\tau) = E[z(t) \cdot z(t+\tau)]$$

$$= [x(t) + y(t)][x(t+\tau) + y(t+\tau)]$$

$$= E[z(t)z(t+\tau)]$$

$$= E[(x(t) + y(t))[x(t+\tau) + y(t+\tau)]]$$

$$= E[x(t)x(t+\tau) + x(t)y(t+\tau) + y(t)x(t+\tau) + y(t)y(t+\tau)]$$

$$= R_{xx}(\tau) + R_{xy}(\tau) + R_{yx}(\tau) + R_{yy}(\tau)$$

$$= E[x(t)x(t+\tau)] + E[x(t)y(t+\tau)] + E[y(t)x(t+\tau)] + E[y(t)y(t+\tau)]$$

$$= R_{xx}(\tau) + R_{xy}(\tau) + R_{yx}(\tau) + R_{yy}(\tau)$$

CROSS-CORRELATION: $(R_{xy}(\tau))$:

The cross-correlation function is defined as the function that gives relation between time delayed version of the one random process and undelayed version of another random process. It is function of delayed (τ) .

$$R_{xy}(\tau) = E[x(t)y(t+\tau)]$$

$$R_{xy}(t_1, t_2) = E[x(t_1)y(t_2)]$$

properties of cross-correlation:

1) cross-correlation satisfy symmetric properties then

$$R_{xy}(\tau) = R_{yx}(\tau)$$

$$R_{xy}(\tau) = E[x(t)y(t+\tau)]$$

$$R_{yx}(\tau) = E[y(t)x(t+\tau)]$$

Replace τ by $-\tau$

$$R_{yx}(-\tau) = E[y(t)x(t-\tau)]$$

$$\text{let } t - \tau = u, \quad t = u + \tau$$

$$= E[y(u+\tau)x(u)]$$

$$= E[x(u)y(u+\tau)]$$

$$\boxed{R_{xy}(\tau) = R_{yx}(\tau)}$$

2) If $x(t)$ and $y(t)$ are two random processes then it satisfies the inequality $|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0) \cdot R_{yy}(0)}$

consider two random processes $x(t)$ and $y(t)$ with auto correlation function $R_{xx}(\tau)$ and $R_{yy}(\tau)$ then it assume the inequality as

$$E\left[\frac{x(t)}{\sqrt{R_{xx}(0)}} \pm \frac{y(t)}{\sqrt{R_{yy}(0)}}\right]^2 \geq 0$$

$$E\left[\frac{x^2(t)}{R_{xx}(0)} \pm \frac{2x(t)y(t)}{\sqrt{R_{xx}(0) \cdot R_{yy}(0)}} + \frac{y^2(t)}{R_{yy}(0)}\right] \geq 0$$

$$E\left[\frac{x^2(t)}{R_{xx}(0)}\right] + E\left[\frac{y^2(t)}{R_{yy}(0)}\right] \pm E\left[\frac{2x(t)y(t)}{\sqrt{R_{xx}(0) \cdot R_{yy}(0)}}\right] \geq 0$$

According to Auto-correlation property

$$E[x^2(t)] = R_{xx}(0)$$

$$\frac{R_{xx}(0)}{R_{xx}(0)} + \frac{R_{yy}(0)}{R_{yy}(0)} \pm \frac{2R_{xy}(\tau)}{\sqrt{R_{xx}(0) \cdot R_{yy}(0)}} \geq 0$$

$$2 \pm \frac{2R_{xy}(\tau)}{\sqrt{R_{xx}(0) \cdot R_{yy}(0)}} \geq 0$$

$$2\left[1 \pm \frac{R_{xy}(\tau)}{\sqrt{R_{xx}(0) \cdot R_{yy}(0)}}\right] \geq 0$$

$$\sqrt{R_{xx}(0) \cdot R_{yy}(0)} \pm R_{xy}(\tau) \geq 0$$

$$\sqrt{R_{xx}(0) \cdot R_{yy}(0)} \geq |R_{xy}(\tau)| \rightarrow \text{Geometric mean}$$

Absolute value of cross-correlation

3) If $x(t)$ and $y(t)$ are two random processes with the autocorrelation function $R_{xx}(\tau)$ and $R_{yy}(\tau)$ then the cross-correlation function satisfies the inequality.

$$|R_{xy}(\tau)| \leq \frac{1}{2}(R_{xx}(0) + R_{yy}(0))$$

consider from the property of geometric mean and arithmetic mean, if $R_{xx}(\tau)$ and $R_{yy}(\tau)$ are the auto-correlation functions, then it satisfies the inequality as,

The geometric mean always less than (or) equal to arithmetic mean

$$\text{i.e. } \sqrt{R_{xx}(0) \cdot R_{yy}(0)} \leq \frac{1}{2}(R_{xx}(0) + R_{yy}(0)) \rightarrow (1)$$

From the property, the absolute value of cross correlation always less than (or) equal to Geometric mean

$$|R_{xx}(\tau)| \leq \sqrt{R_{xx}(0) \cdot R_{yy}(0)} \rightarrow (2)$$

From (1) and (2),

$$|R_{xy}(\tau)| \leq \frac{1}{2}(R_{xx}(0) + R_{yy}(0))$$

4) If $x(t)$ and $y(t)$ are the random processes which are statistically independent at least wide sense stationary random process $R_{xy}(\tau) = \bar{x} \cdot \bar{y}$

Proof: Let the Random process $x(t)$ and $y(t)$ then its cross-correlation function is given as

$$R_{xy}(\tau) = E[x(t)y(t+\tau)]$$

$$= E[x(t)y(t)]$$

$$= E[x(t)]E[y(t)]$$

$$= \bar{x} \cdot \bar{y}$$

$$R_{xy}(\tau) = \bar{x} \cdot \bar{y}$$

5) If two random processes $x(t)$ and $y(t)$ have zero mean and at least wide sense stationary process then

$$\lim_{\tau \rightarrow \infty} R_{xy}(\tau) = 0$$

(I) According to autocorrelation property,

$$\lim_{T \rightarrow \infty} R_{xx}(\tau) = \bar{x}^2$$

Where, Mean value = 0

$$\lim_{T \rightarrow \infty} R_{xy}(\tau) = 0$$

(II) From cross-correlation property,

$$R_{xy}(\tau) = \bar{x} \cdot \bar{y}$$

$$\lim_{T \rightarrow \infty} R_{xy}(\tau) = \bar{x} \cdot \bar{y} = 0$$

* Auto-correlation function of an ergodic stationary process

With no periodic component is given as $100 + \frac{4}{1+6\tau^2}$

Find the Mean and Variance.

Sol: Given that, $R_{xx}(\tau) = 100 + \frac{4}{1+6\tau^2}$

WKT, $\lim_{T \rightarrow \infty} R_{xx}(\tau) = \bar{x}^2$

$$\lim_{T \rightarrow \infty} \left[100 + \frac{4}{1+6\tau^2} \right] = \bar{x}^2$$

$$100 = \bar{x}^2$$

Mean value, $\boxed{\bar{x} = 10}$

$$\sigma_x^2 = E[x^2(t)] - [E[x]]^2 \rightarrow (*)$$

WKT, $E[x^2(t)] = R_{xx}(0)$

$$\lim_{\tau \rightarrow 0} \left[100 + \frac{4}{1+6\tau^2} \right] = E[x^2(t)]$$

$$100 + 4 = E[x^2(t)]$$

$$\boxed{E[x^2(t)] = 104}$$

from (*),

$$\sigma_x^2 = 104 - 100 = 4$$

$$\boxed{\sigma_x^2 = 4}$$

* The Autocorrelation function for a stationary Ergodic function with no periodic component is $R_{xx}(\tau) = 625 + \frac{16}{1+36\tau^2}$
Find mean and variance.

Sol:

Given that,

$$R_{xx}(\tau) = 625 + \frac{16}{1+36\tau^2}$$

$$\lim_{\tau \rightarrow \infty} R_{xx}(\tau) = \bar{x}^2$$

$$\lim_{\tau \rightarrow \infty} \left[625 + \frac{16}{1+36\tau^2} \right] = \bar{x}^2$$

$$625 + 0 = \bar{x}^2$$

$$\bar{x}^2 = 625$$

$$\boxed{\bar{x} = 25}$$

$$\text{Variance } \sigma_x^2 = E[x^2(t)] - E[x(t)]^2$$

$$\text{WKT, } R_{xx}(0) = E[x^2(t)] \quad E[x(t)] = R_{xx}(0)$$

$$\sigma_x^2 = 625 + \frac{16}{1+36(0)} - 625$$

$$= 625 + 16 - 625$$

$$= 641 - 625$$

$$= 16$$

$$\boxed{\sigma_x^2 = 16}$$

* A Random process is given by $x(t) = A \cos(\omega_0 t + \theta)$ is wide sense stationary. If it is assumed that ω_0 is constant and θ is uniformly distributed variable in the interval $[0, 2\pi]$.

Sol. Given that,

$$x(t) = A \cos(\omega_0 t + \theta)$$

where, A, ω_0 are constants and θ is uniformly distributed variable over $(0, 2\pi)$.

If, $E[x(t)]$ is constant and $R_{xx}(\tau)$ is independent of time then, given system is wide-sense stationary process.

θ is uniformly distrib

$$f_{\theta}(\theta) = \frac{1}{b-a}, \quad a < \theta < b$$

$$f_{\theta}(\theta) = \frac{1}{2\pi - 0} = \frac{1}{2\pi}, \quad 0 < \theta < 2\pi$$

$$\boxed{f_{\theta}(\theta) = \frac{1}{2\pi}}$$

(i) $E[x(t)] = \text{constant}$

$$E[A \cos(\omega_0 t + \theta)] = \int_{-\infty}^{\infty} A \cos(\omega_0 t + \theta) f_{\theta}(\theta) d\theta$$

$$= \int_0^{2\pi} A \cos(\omega_0 t + \theta) \frac{1}{2\pi} d\theta$$

$$= \frac{A}{2\pi} \int_0^{2\pi} A \cos(\omega_0 t + \theta) d\theta$$

$$= \frac{A}{2} \left[\sin(\omega_0 t + \theta) \right]_0^{2\pi}$$

$$= \frac{A}{2} [\sin(\omega_0 t + 2\pi) - \sin(\omega_0 t + 0)]$$

$$= \frac{A}{2} [0 - 0]$$

(ii) $R_{xx}(\tau) \rightarrow$ Independent of time

$$R_{xx}(\tau) = E[x(t)x(t+\tau)]$$

$$= E[A \cos(\omega_0 t + \theta) A \cos(\omega_0 (t+\tau) + \theta)]$$

$$= E[A^2 \cos(\omega_0 t + \theta) \cos(\omega_0 (t+\tau) + \theta)]$$

$$= E\left[\frac{A^2}{2} \cos[\omega_0 t + \theta + \omega_0 (t+\tau) + \theta] + \cos[\omega_0 t + \theta - \omega_0 (t+\tau) - \theta]\right]$$

$$= E\left[\frac{A^2}{2} \cos[\omega_0 (2t+\tau) + 2\theta] + \cos[\omega_0 \tau]\right]$$

$$= \int_{-\infty}^{\infty} \frac{A^2}{2} \cos(\omega_0 (2t+\tau) + 2\theta) + \cos \omega_0 \tau \frac{1}{2\pi} d\theta$$

$$= \frac{A^2}{2} \int_0^{2\pi} [\cos(\omega_0 (2t+\tau) + 2\theta) + \cos \omega_0 \tau] \frac{1}{2\pi} d\theta$$

$$= \frac{A^2}{4\pi} \left[\frac{\sin[\omega_0 (2t+\tau) + 2\theta]}{2} + \cos \omega_0 \tau \sin \theta \right]_0^{2\pi}$$

$$= \frac{A^2}{4\pi} [\sin 0 + \cos \omega_0 \tau (\sin 2\pi - \sin 0) + \cos \omega_0 \tau (2\pi - 0)]$$

$$= \frac{A^2}{4\pi} [\cos \omega_0 \tau (2\pi)]$$

$$= \frac{\omega_0 A^2}{4\pi} \frac{2\pi \cos \omega_0 \tau (A^2)}{4\pi \cdot 2}$$

$$= \frac{A^2 \omega_0}{4\pi} = A^2 \cos \omega_0 \tau$$

Auto co-variance:

consider two random processes $x(t)$ and $x(t+\tau)$ at time intervals t and $t+\tau$. The auto co-variance function can be represented as

$$C_{xx}(t, t+\tau) = E[x(t) - E[x(t)]] [x(t+\tau) - E[x(t+\tau)]]$$

$$C_{xx}(t, t+\tau) = R_{xx}(t, t+\tau) - E[x(t)] E[x(t+\tau)]$$

cross co-variance:

If two random processes $x(t)$ and $y(t)$ have random variables $x(t)$ and $y(t+\tau)$ then cross-co-variance can be defined as

$$C_{xy}(t, t+\tau) = E[x(t) - E[x(t)]] [y(t+\tau) - E[y(t+\tau)]]$$

Gaussian Random process:

consider a continuous random process $x(t)$. let N random variables be defined as

$x_1 = x(t_1)$; $x_2 = x(t_2)$; ... $x_N = x(t_N)$ be defined at time instants $t_1, t_2, \dots, t_N$.

If these random variables are jointly Gaussian for

$N=1, 2, 3, \dots, N$ at time instants $t = t_1, t_2, t_3, \dots, t_N$

then the random process $x(t)$ is called as Gaussian

Random process, then the joint probability density function for Gaussian Random variables is given as

$$f_x(x_1, x_2, x_3, \dots, x_N; t_1, t_2, \dots, t_N) = \frac{e^{-\frac{1}{2}(x-\bar{x})^T [\sigma_x]^{-1} (x-\bar{x})}}{\sqrt{(2\pi)^N |\sigma_x|}}$$

Poisson Random process:

I_t is a discrete random process which represents no. of times that some event has occurred as a function of time.

* In these functions a single event occurs at random time.

Note:

1. $x(0) = 0$
2. Only one event is occurred at one instant of time.

Random process is defined in time domain. This unit defines the important concept of characterizing random process in frequency domain. These all characterizations are spectral characterizations. To consider a random process $x(t)$, the amplitude of random process, where it varies randomly with time cannot satisfy Dirichlet conditions. Therefore, it is not possible to apply Fourier transform directly on the random process. For the frequency domain analysis, thus, the auto correlation function of wide sense stationary (WSS) random process used to study the spectral characteristics such as power density spectrum (PSD) power spectral density. Fourier transformation of $x(t)$.

$$F[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

→ Inverse Fourier transform of $X(\omega)$.

$$F^{-1}[X(\omega)] = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

→ The spectral characteristics gives the relation in Fourier transform to the

there, the Fourier transform of Autocorrelation function

$R_{xx}(\tau)$ is defined as power spectral density (PSD)

UNIT-5 RANDOM PROCESS AND SPECTRAL CHARACTERISTICS

In this unit, we will study the characteristics of Random process regarding correlation and covariance function which is defined in time domain. This unit explores the important concept of characterizing Random process in frequency domain. These all characteristics are spectral characteristics.

→ consider a Random process $x(t)$, the amplitude of random process, when it varies randomly with time does not satisfy Dirichlet's conditions. Therefore, it is not possible to apply Fourier transform directly on the Random process for the frequency domain analysis. Thus, the autocorrelation function of wide sense-stationary (WSS) Random process is used to study the spectral characteristics such as power density spectrum (PDS) power spectral density.

→ Fourier transformation of $x(t)$,

$$F[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

→ Inverse Fourier transform of $X(\omega)$,

$$F^{-1}[X(\omega)] = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

→ The spectral characteristics gives the relation in Fourier transform to the time

Hence, the Fourier transform of Autocorrelation function

$R_{xx}(\tau)$ is defined as power spectral density (PDS) power density spectrum ($S_{xx}(\omega)$)

→ The Relation between power spectral density and Autocorrelation function is

$$\text{FT} [R_{xx}(\tau)] = S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$$

→ Inverse fourier transform of Autocorrelation is

$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} d\omega$$

→ If $x(t)$ is a Random process then power spectral density can also be defined as $S_{xx}(\omega)$, where

$$S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{E[|x_T(\omega)|^2]}{2T}$$

Note:

Hence, the Autocorrelation function $R_{xx}(\tau)$ and power spectral density $S_{xx}(\omega)$ are fourier transform pairs.

Properties of power spectral density:

1) The property of power spectral density $S_{xx}(\omega)$ a wide sense stationary random process is given by $S_{xx}(\omega) \geq 0$.

Proof: from the definition of power spectral density,

$$S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{E[|x_T(\omega)|^2]}{2T}$$

is always non-negative real number hence, the power spectral density is always positive quantity constant.

2) The power spectral density at zero frequency is equal to the area under curve for autocorrelation function.

Proof: From the definition of Autocorrelation function,

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$$

At $\omega = 0$

$$S_{xx}(0) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j(0)\tau} d\tau$$

$$S_{xx}(0) = \int_{-\infty}^{\infty} R_{xx}(\tau) d\tau$$

3) The power density spectrum of Random process $x(t)$ is an even function. then $S_{xx}(\omega) = S_{xx}(-\omega)$

proof:

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$$

$$S_{xx}(-\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{j\omega\tau} d\tau$$

Replace τ by $-\tau$,

$$S_{xx}(-\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau \quad [\because R_{xx}(-\tau) = R_{xx}(\tau)]$$

$$= S_{xx}(\omega)$$

$$S_{xx}(-\omega) = S_{xx}(\omega)$$

4) The power spectral density of Random process $x(t)$ is always real function.

proof:

WKT,

$$S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

Where, due to mod (11), and squaring, it always remains a real number.

5) If $S_{xx}(\omega)$ is a power spectral density for a wide sense stationary Random process $x(t)$ then

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega = E[|x(t)|^2] = R_{xx}(0)$$

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$$

$$\text{WKT, } R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} d\omega$$

Proof:

Consider a wide sense stationary Random process $x(t)$ and we know that the time average of auto-correlation function is given by,

$$A[R_{xx}(\tau)] = A[E[x(t)x(t+\tau)]] \rightarrow (1)$$

It is a wide sense stationary Random process, so

$$x(t) = x(t+\tau),$$

from (1),

$$\therefore A[R_{xx}(\tau)] = AE[x^*(t)]$$

From the definition (2),

$$S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{E[|x_T(\omega)|^2]}{2T} \rightarrow (2)$$

WKT,

$$|x_T(\omega)|^2 = x_T(\omega) \cdot x_T^*(\omega) \rightarrow (3)$$

$$x_T(\omega) = \int_{-T}^T x(t_1) e^{-j\omega t_1} dt_1 \rightarrow (4)$$

$$x_T^*(\omega) = \int_{-T}^T x(t_2) e^{+j\omega t_2} dt_2 \rightarrow (5)$$

substitute 4, 5 in (2)

$$S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{E[x_T(\omega) \cdot x_T^*(\omega)]}{2T}$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} E \left[\int_{-T}^T x(t_1) e^{-j\omega t_1} dt_1 \int_{-T}^T x(t_2) e^{+j\omega t_2} dt_2 \right]$$

$$S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} E \left[\int_{-T}^T \int_{-T}^T x(t_1) x(t_2) e^{-j\omega(t_1 - t_2)} dt_1 dt_2 \right]$$

WKT,

Above random process satisfies wide-sense stationary

$$S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} E \left| \int_{-T}^T \int_{-T}^T x(t_1) x(t_2) e^{-j\omega(t_2 - t_1)} dt_1 dt_2 \right|$$

$$|t_2 - t_1| = \tau$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} E \left| \int_{-T}^T \int_{-T}^T x(t+\tau) x(t) e^{-j\omega(t+\tau-t)} dt d(t+\tau) \right|$$

$$S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} E \left| \int_{-T}^T \int_{-T}^T R_{xx}(\tau) e^{-j\omega\tau} dt d\tau \right|$$

Applying Inverse Fourier transform on both sides,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega t} d\omega = \lim_{T \rightarrow \infty} \frac{1}{2T} E \left| \int_{-\infty}^{\infty} \int_{-T}^T \int_{-T}^T R_{xx}(\tau) e^{-j\omega\tau} e^{j\omega t} dt d\tau d\omega \right|$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} E \left| \int_{-\infty}^{\infty} \int_{-T}^T \int_{-T}^T R_{xx}(\tau) e^{-j\omega(\tau-t)} dt d\tau d\omega \right|$$

$$= \frac{1}{2T} E \left| \int_{-T}^T \int_{-T}^T R_{xx}(\tau) \int_{-\infty}^{\infty} e^{-j\omega(\tau-t_1+t)} d\omega dt d\tau \right|$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} E \left| \int_{-T}^T \int_{-T}^T R_{xx}(\tau) dt \int_{-\infty}^{\infty} \delta(\tau - t_1 - t) dt_1 \right|$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} E \left| \int_{-T}^T (R_{xx}(\tau)) d\tau \int_{-\infty}^{\infty} \delta(\tau - t_1 - t) dt \right|$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} E \left| \int_{-T}^T R_{xx}(\tau) d\tau \left[\int_{-\infty}^{\infty} \delta(\tau - t_1 - t) dt = 1 \right] \right|$$

$$S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} E \left| \int_{-T}^T R_{xx}(\tau) d\tau \right|$$

$$\therefore S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} E \left| \int_{-T}^T R_{xx}(t_1, t) dt \right|$$

Cross power spectral density:

consider a real random process $x(t)$ and $y(t)$ are jointly wide sense stationary process then the cross spectral density is defined as Fourier transform of cross correlation function $[of R_{xy}(\tau)]$ of $x(t)$ and $y(t)$ mathematically represented as

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j\omega\tau} d\tau$$

$$S_{yx}(\omega) = \int_{-\infty}^{\infty} R_{yx}(\tau) e^{-j\omega\tau} d\tau$$

→ Inverse fourier transform of cross power spectral density.

$$R_{xy}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega) e^{j\omega\tau} d\omega$$

$$R_{yx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{yx}(\omega) e^{j\omega\tau} d\omega$$

properties of cross power spectral density:

Definition 2: consider two random processes $x(t)$ and $y(t)$ if $x_t(\omega)$ and $y_t(\omega)$ are fourier transform of $x(t)$ and $y(t)$ then cross power spectral density is defined as

$$S_{xy}(\omega) = \lim_{T \rightarrow \infty} \frac{E |x_T^*(\omega) y_T(\omega)|}{2T}$$

$$S_{yx}(\omega) = \lim_{T \rightarrow \infty} \frac{E |y_T^*(\omega) x_T(\omega)|}{2T}$$

$$1. S_{xy}(\omega) = S_{yx}(-\omega) = S_{yx}^*(\omega)$$

Proof: $S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j\omega\tau} d\tau$

τ is replaced by $-\tau$

$$= \int_{-\infty}^{\infty} R_{xy}(-\tau) e^{-j\omega(-\tau)} d\tau$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j(-\omega)\tau} d\tau$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j\omega^*\tau} d\tau$$

$$S_{xy}(\omega) = S_{xy}^*(\omega)$$

$$S_{xy}(\omega) - S_{xy}^*(\omega) = S_{xy}(-\omega)$$

2) The real part of cross power spectral density i.e. real part of $\text{Re}[S_{xy}(w)]$, $\text{Re}[S_{yx}(w)]$ are even-function.

proof:

$$e^{-j\omega\tau} = \underbrace{\cos\omega\tau}_{\text{Real}} - j\underbrace{\sin\omega\tau}_{\text{imaginary}}$$

$$\text{Re}[S_{xy}(w)] = \int_{-\infty}^{\infty} R_{xy}(\tau) \cos\omega\tau d\tau$$

Replace w by $-w$

$$\begin{aligned}\text{Re}[S_{xy}(-w)] &= \int_{-\infty}^{\infty} R_{xy}(\tau) \cos(-w\tau) d\tau \\ &= \int_{-\infty}^{\infty} R_{xy}(\tau) \cos w\tau d\tau\end{aligned}$$

$$\boxed{\text{Re}[S_{xy}(-w)] = \text{Re}[S_{xy}(w)]}$$

Similarly,

$$\text{Re}[S_{yx}(w)] = \int_{-\infty}^{\infty} R_{yx}(\tau) \cos\omega\tau d\tau$$

Replace w by $-w$

$$\begin{aligned}\text{Re}[S_{yx}(-w)] &= \int_{-\infty}^{\infty} R_{yx}(\tau) \cos(-w\tau) d\tau \\ &= \int_{-\infty}^{\infty} R_{yx}(\tau) \cos w\tau d\tau\end{aligned}$$

$$\boxed{\text{Re}[S_{yx}(-w)] = \text{Re}[S_{yx}(w)]}$$

3) The imaginary part of $S_{xy}(w)$ and imaginary part of the function $S_{yx}(w)$ are odd functions of

proof:

$$S_{xy}(w) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j\omega\tau} d\tau$$

$$e^{-j\omega\tau} = \cos\omega\tau - j\sin\omega\tau$$

$$\text{Im}[S_{xy}(w)] = \int_{-\infty}^{\infty} R_{xy}(\tau) (-\sin\omega\tau) d\tau$$

$$= - \int_{-\infty}^{\infty} R_{xy}(\tau) \sin \omega \tau d\tau$$

Replace ω by $-\omega$, then

$$\text{Im}[S_{xy}(-\omega)] = - \int_{-\infty}^{\infty} R_{xy}(\tau) \sin(-\omega) \tau d\tau$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau) \sin \omega \tau d\tau$$

$$\text{Im}[S_{xy}(-\omega)] = \text{Im}[S_{xy}(\omega)] //$$

Similarly,

$$S_{yx}(\omega) = \int_{-\infty}^{\infty} R_{yx}(\tau) \sin(-\omega) \tau d\tau$$

$$\text{Im}[S_{yx}(\omega)] \cdot e^{-j\omega\tau} = \cos \omega \tau - j \sin \omega \tau$$

$$\text{Im}[S_{yx}(\omega)] = \int_{-\infty}^{\infty} R_{yx}(\tau) (-\sin \omega \tau) d\tau$$

$$= - \int_{-\infty}^{\infty} R_{yx}(\tau) \sin \omega \tau d\tau$$

Replace ω by $-\omega$,

$$\text{Im}[S_{yx}(-\omega)] = - \int_{-\infty}^{\infty} R_{yx}(\tau) \sin(-\omega) \tau d\tau$$

$$= \int_{-\infty}^{\infty} R_{yx}(\tau) \sin \omega \tau d\tau$$

$$\text{Im}[S_{yx}(-\omega)] = \text{Im}[S_{yx}(\omega)] //$$

4. $S_{xy}(\omega) = 0$, $S_{yx}(\omega) = 0$ is the random process of $x(t)$ and $y(t)$ are orthogonal.

proof: $S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{j\omega\tau} d\tau$

$$= \int_{-\infty}^{\infty} 0 \cdot e^{j\omega\tau} d\tau$$

$$= 0 \quad [\because \text{Orthogonal to each other}]$$

$$\therefore \boxed{S_{xy}(\omega) = 0}$$

$$S_{yx}(\omega) = \int_{-\infty}^{\infty} R_{yx}(\tau) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} 0 \cdot e^{-j\omega\tau} d\tau$$

$$= 0$$

$$\therefore \boxed{S_{yx}(\omega) = 0}$$

5. If $x(t)$ and $y(t)$ are uncorrelated having constant mean values. The cross power spectral density is given by $S_{xy}(\omega) = 2\pi \bar{x} \cdot \bar{y} \delta(\omega)$

Proof:

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} E[x(t)y(t)] e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} E[x(t)] E[y(t)] e^{-j\omega\tau} d\tau$$

$$= \bar{x} \cdot \bar{y} \int_{-\infty}^{\infty} e^{-j\omega\tau} d\tau$$

$$= \bar{x} \cdot \bar{y} 2\pi \delta(\omega)$$

$$= 2\pi \bar{x} \cdot \bar{y} \delta(\omega)$$

$$\therefore \boxed{S_{xy}(\omega) = 2\pi \bar{x} \cdot \bar{y} \delta(\omega)}$$

**
6) Write the relation between cross power spectral density and crosscorrelation.

Let $x_T(t)$ & $y_T(t)$ are ensemble numbers of $x(t)$ and $y(t)$ respectively.

WKT, cross power spectral density,

$$S_{xy}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} E[x_T^*(\omega) y_T(\omega)]$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt \quad \boxed{0 = (\omega)_{pxc}}$$

$$x_T^*(\omega) = \int_{-T}^T x(t) e^{-j\omega t} dt$$

$$y_T(\omega) = \int_{-T}^T y(t) e^{-j\omega t} dt$$

$$S_{xy}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} E \left| \int_{-T}^T x(t) e^{j\omega t} dt \int_{-T}^T y(t) e^{-j\omega t} dt \right|$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} E \left| \int_{-T}^T \int_{-T}^T x(t) y(t) e^{j\omega t} e^{-j\omega t_1} dt dt_1 \right|$$

Applying Laplace transform on both sides,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega) e^{j\omega \tau} d\omega = \lim_{T \rightarrow \infty} \frac{1}{2T} \frac{1}{2\pi} E \int_{-\infty}^{\infty} \int_{-T}^T \int_{-T}^T x(t) y(t) e^{-j\omega(t-t_1)} e^{j\omega \tau} dt dt_1 d\omega$$

$$= \frac{1}{2T} \lim_{T \rightarrow \infty} \frac{1}{2\pi} E \left| \int_{-\infty}^{\infty} \int_{-T}^T \int_{-T}^T x(t) y(t) e^{-j\omega(t-t_1-\tau)} dt dt_1 d\omega \right|$$

WKT, $\int_{-\infty}^{\infty} e^{-j\omega \tau} d\omega = 2\pi \delta(t_1 - t - \tau)$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \frac{1}{2\pi} E \left| \int_{-T}^T \int_{-T}^T x(t) y(t) 2\pi \delta(t_1 - t - \tau) dt dt_1 \right|$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} E \left| \int_{-T}^T x(t) y(t) dt \int_{-T}^T \delta(t_1 - t - \tau) dt_1 \right|$$

WKT, $\int_{-T}^T \delta(t_1 - t - \tau) dt_1 = 1$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} E \left| \int_{-T}^T x(t) y(t) dt \right|$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T R_{xy}(t, t) dt$$

$$\boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega) e^{j\omega \tau} d\omega = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T R_{xy}(t, t) dt}$$

7. Derive the expression for power spectral density on random process (8) Write short notes on power spectral density (8) Relation b/w power and power spectral density.

sd: $S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$

$$S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{E |x_T(\omega)|^2}{2T}$$

$$x(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x_T(\omega) = \int_{-T}^T x(t) e^{-j\omega t} dt$$

Power = $\frac{1}{2T} \times \text{energy of } x(t)$

$$\text{Energy} = \int_{-\infty}^{\infty} x^2(t) dt$$

$$= \int_{-T}^T x^2(t) dt$$

$$= \int_{-T}^T x(t) x(t) dt$$

$$x(t) = \frac{1}{2\pi} \int_{-T}^T x(\omega) e^{j\omega t} d\omega$$

$$\text{Energy} = \frac{1}{2\pi} \int_{-T}^T x(\omega) e^{j\omega t} d\omega \int_{-T}^T x(t) dt$$

$$= \frac{1}{2\pi} \int_{-T}^T x(t) e^{j\omega t} dt \int_{-T}^T x(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-T}^T x(-\omega) x(\omega) d\omega \quad \left[\because \int_{-T}^T x(t) e^{j\omega t} dt = x(-\omega) \right]$$

$$\text{Energy} = \frac{1}{2\pi} \int_{-T}^T |x(\omega)|^2 d\omega$$

$\therefore \text{power} = \frac{1}{2T} \times \text{Energy of } x(t)$

$$= \frac{1}{2T} \cdot \frac{1}{2\pi} \int_{-T}^T |x(\omega)|^2 d\omega$$

The power random process of $x(t)$ over the interval $-\infty < t < \infty$ is expectation of $P(t)$ has $T \rightarrow \infty$

$$E[P(t)] = P_{xx} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(\omega)|^2 d\omega$$

$$P_{xx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega$$

$$P_{xx} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(\omega)|^2 d\omega$$

Power spectral density

Power spectral density

Consider two random variables x and y then joint moment about origin is defined as the expected value of $x^k y^k$, mathematically it can be expressed as

$$M_{xy} = E[x^k y^k] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^k y^k p_{xy}(x, y) dx dy$$

Joint moment

If $x = 0, y = 0$

$$M_{xx} = E[x^2] = \int_{-\infty}^{\infty} x^2 p_x(x) dx$$

$$= \int_{-\infty}^{\infty} x^2 p_x(x) dx$$

$$= \int_{-\infty}^{\infty} x^2 p_x(x) dx$$

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First order moment

If $x = 0, y = 0$

$$M_{xx} = E[x^2] = \int_{-\infty}^{\infty} x^2 p_x(x) dx$$